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Quasiperiodic Renormalisation

Quasiperiodic sums, products and composition sum operators

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A thesis submitted for the degree of

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Abstract

We study two topics in the renormalisation group theory of quasiperiodic systems, both topics having a number of important applications.

Firstly we apply renormalisation techniques to a family of functions we designate quasiperiodic sums and products. These functions link the study of critical phenomena in diverse fields such as the emergence of strange non-chaotic attractors, critical KAM theory, convergence of ergodic averages, and q-series (much used in string theory). In pure mathematics they have also been studied extensively in complex power series analysis, partition theory, and Diophantine approximation.

We set these currently dispersed results within a unified framework, and develop a more systematic approach to three key examples. We improve significantly on a series of number-theoretic results obtained by Sierpinsky, Hardy, Hecke, Lang et al; we provide a rigorous proof of some empirically obtained results recently reported by Knill et al (2011-12); and we settle (negatively) an open question of Erdős-Szekeres-Lubinsky dating initially from 1959.

Secondly we study the fixed points of quasiperiodic renormalisation. These arise in the study of the Harper equation (almost Mathieu equation), barrier billiards, and a number of other scenarios. We identify the natural setting for their study as a certain class of linear operators (composition sum operators) acting on unbounded (non-Banach) function spaces. We develop the necessary foundations for this theory, and then apply it to construct the space of all fixed points of the golden renormalisation whose singularities are either poles, essential singularities, or logarithmic singularities of a certain simple type. This fixed point space is shown to include previously unknown fixed points.

Acknowledgements

I am grateful to my two co-supervisors, Dr Robert Hasson and Professor Andrew Osbaldestin, for their helpful comments and suggestions both during the time of my PhD project and on this document.

Heartfelt thanks go to my wife Jane for endless support, understanding, and camellia sinensis, and without whose encouragement this whole project would never have started.

I owe incalculable thanks to my primary supervisor Dr Ben Mestel for his unstinting generosity of time, ideas and spirit. His enthusiasm transformed every supervision from a chore into a joy (a priceless skill). His influence is densely distributed and decidedly non-meagre within this work!

Finally, this work is dedicated to my daughter Sara, with whom I am blessed to share my love of mathematics.

Declaration

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis or below.

The following parts of the thesis have led to joint papers with my supervisor Ben Mestel, and with myself as the corresponding author.

Much of chapter 4 was published in the Journal of Mathematical Analysis and Applications[57], and was the principal responsibility of the author, with supervisory and editorial contributions by my co-author.

Much of chapter 6 was published in the Journal of Difference Equations and Applications[58], and was very much a collaborative effort but based substantially on the author's original research.

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Chapter 1

Introduction

In this work we will develop and extend theory, techniques and results in two areas of the renormalisation group theory of quasiperiodic systems. Both of these areas are of interest in their own right, but also have multiple applications.

The first area is that of quasiperiodic sums and products. These arise in a surprising number of areas of mathematics and physics, and have been studied extensively within individual disciplines. However these studies have been somewhat fragmented, kept separate to an extent by the differing terminologies and notations of the various disciplines. We will try to show the underlying unity of the subject, and the benefits which flow from studying it as such. The mathematical structures are straightforward to define and compute, but exhibit beautiful and intriguing fractal characteristics which gives them intrinsic interest. We will start in this introduction with definitions, examples, and a brief discussion of their applications. We continue in chapters 2–5 with a detailed study of three key examples. We will build increasingly on renormalisation techniques during the course of these chapters.

The second area is that of the fixed point theory of composition sum operators, a class of linear operators on unbounded (non-Banach) function spaces. This topic is motivated by the desire to study the fixed points of quasiperiodic renormalisation group operators. We use the theory to find and construct fixed points of the golden renormalisation group, and to show that this is the complete set of solutions whose singularities are poles, essential singularities, and certain types of logarithmic singularities (PESL singularities - see 6.4.2). .

1.1 Initial definitions and examples

1.1.1 Quasiperiodic sums and products

Definitions

Quasiperiodic sums and products have a simple conceptual interpretation as operations over fixed rotations of the circle. However we will find it notationally simpler to provide a formal definition in slightly more abstract terms, and then to see how the interpretation can be made.

We first recall the function $\{.\} : \mathbb{R} \to [0, 1)$, called the remainder function, or fractional part function, which maps $x \in \mathbb{R}$ to the unique $\{x\} \in [0, 1)$ satisfying $x - \{x\} \in \mathbb{Z}$.

Definition 1.1.1. Let V be a semi-group under the operations '+' and/or '×'. For $f : [0, 1) \to V$ and $x, \alpha \in \mathbb{R}$ we define for each $n \in \mathbb{N}$ the nth QUASIPERIODIC SUM and/or QUASIPERIODIC PRODUCT:

$$S_n(x, \alpha, f) = \sum_{k=0}^{n-1} f(\{x + k\alpha\}) \tag{1.1}$$

$$P_n(x, \alpha, f) = \prod_{k=0}^{n-1} f(\{x + k\alpha\}) \tag{1.2}$$

The function f is called the VALUE FUNCTION.

Notes:

1. Algebraically S_n and P_n are really identical constructs. To avoid overmuch repetition, when we are talking generally about quasiperiodic sums or the function S_n, we mean to include the possibility of products or P_n respectively.

2. We will simplify the notation when appropriate, by writing $S_n(x, \alpha)$ when f is clear.

3. The special case of $x = \alpha$ is important. We will call it the BASE CASE, and designate it $S_n(\alpha) = S_n(\alpha, \alpha) = \sum_{k=1}^{n} f(\{n\alpha\})$.

4. By regarding $[0, 1)$ as the coordinate system for a circle and noting $S_n(x, \alpha, f) = S_n(\{x\}, \{\alpha\}, f)$ we can interpret S_n as a sum of the values of f over the first n points of the orbit of the point $\{x\}$ of the circle under a constant rotation by α circle revolutions.

5. Care must be taken over the remainder function in the definition in order to avoid subtle problems. For example the value function $f(x) = \sin \pi x$ gives rise to the sum of terms $\sin \pi\{x + k\alpha\}$ which are always positive, but the terms $\sin \pi(x + k\alpha)$ can take negative values.

6. Usually we set $V = \mathbb{R}$, although other cases are possible. We will only study the case $V = \mathbb{R}$ in this document, noting in this case that quasiperiodic sums and products are both defined.

Our main interest is to understand the behaviour of quasiperiodic sums as n grows.

Remark 1.1.2. We will normally use the term QUASIPERIODIC in contradistinction to the term PERIODIC, meaning the value of α is taken to be irrational rather than rational. However sometimes, when there is little risk of ambiguity, it is also very convenient to use it homonymically as an inclusive term which covers both rational and irrational α.

Examples

We now give some simple but important examples of quasiperiodic sums, each of which has been studied in several papers. Note that the differentiator between the examples is just the choice of the value function f, although the examples originate from different disciplines.

In order to develop some intuition we will exhibit some graphs of each function showing the effects of increasing n. Given f each quasiperiodic sum is a function of two variables: we will graph only the special case $S_n(\omega) = S_n(\omega, \omega)$ where ω is the distinguished value known as the GOLDEN ROTATION $\omega = \frac{1}{2}\left(\sqrt{5} - 1\right)$. Note that the golden *rotation* is the fractional part of the golden *ratio*, ie $\omega = \left\{\frac{1}{2}\left(\sqrt{5} + 1\right)\right\}$.

Since our main interest is in the growth of S_n with n, we show a series of graphs with n ranging from 1 to F_m where F_m is a Fibonacci number (the significance of this choice will emerge from the work we will do later).This allows the emergent fractal properties of the graphs to be seen clearly.

1.1.1.1 The sum of remainders (1909)

This is perhaps the simplest possible example of a quasiperiodic sum. If we take the identity function as our value function, the resulting quasiperiodic sum is $S_n(x, \alpha) = \sum_{k=0}^{n-1}\{x + k\alpha\}$. However the terms $\{x + k\alpha\}$ are known to be equidistributed in $[0, 1)$ around the mean of $1/2$. It is more useful therefore to factor out from this sum the linear growth rate of $n/2$, so that we can focus on the fluctuations from the mean. This is equivalent to using a value function $f(y) = y - \frac{1}{2}$, and gives us the quasiperiodic sum $S_n(x, \alpha) = \sum_{k=0}^{n-1}\left(\{x + k\alpha\} - \frac{1}{2}\right)$, known classically as the sum of remainders.

We study this sum in detail in chapter 2. See Fig 1.1.1 for the graph.

The first result on this sum to be published with proof was by Sierpinski (1909)[51][1] who showed that for any irrational α, $S_n(\alpha) = S_n(\alpha, \alpha) = o(n)$.

1.1.2 The Sudler product of sines (1964)

This example uses as its value function perhaps the simplest circle function, namely $f(y) = 2\sin\pi y$. This gives us the quasiperiodic product $P_n(x, \alpha) = \prod_{k=0}^{n-1} 2\sin\pi(\{x + k\alpha\})$.

[1] Lerch first drew attention to the sum in 1904[39] but appears not to have provided proofs of any results.

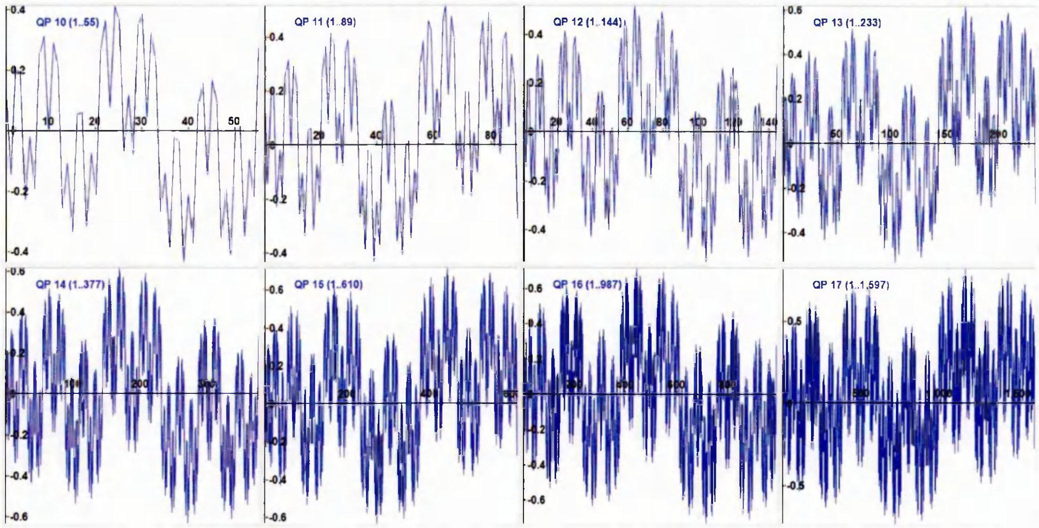

Figure 1.1.1: Graphs of the sum of remainders $S_n(\omega) = \sum_{k=1}^{n}\{k\omega\} - \frac{1}{2}$ against n for $n = 1\ldots F_m$ and $m = 10\ldots 17$. Each graph is the equivalent of inverting (and rescaling) the initial segment $[0, \omega]$ of its successor. The graphs can each be divided into 3 sections - a left and right section of width ω^2 (in renormalised coordinates), and a central section of width ω^3. (Note that $2\omega^2+\omega^3 = 1$). The left section ($[0, \omega^2]$) is a scaled version of the whole; the right section ($[\omega, 1]$) is a scaled (but slightly deformed) version of the whole; the central body is a scaled and inverted version of the whole; the left and centre sections together also form a scaled version of the previous graph; and each graph is an inversion of its neighbours.

The first result on this sum to be published with proof was by Sudler in 1964[55] who obtained an elegant result for $\sup_\alpha P_n(\alpha, \alpha)$ and showed it had exponential growth (see chapter 4 for details).

We study this product in detail in chapters 4 and 5. See Fig 1.1.2 for the graph.

1.1.2.1 The Knill sum of cotangents (2012)

This example uses as its value function the unbounded circle function $f(x) = \cot \pi x$. This gives the quasiperiodic sum $S_n(x, \alpha) = \sum_{k=0}^{n-1} \cot \pi\{x + k\alpha\}$.

This function was first studied by Knill in 2012[31] who established its self-similarity amongst other results (see chapter 3 for further details).

We will study it in detail in chapter 3. See Fig 1.1.3 for the graph.

1.1.3 Quasiperiodic renormalisation

In general, renormalisation describes a class of techniques with the common purpose of analysing scaling symmetries. Important applications have been developed in Quantum Field Theories, Statistical Physics, and Fractal Geometry. Our interest here begins with the latter: the chaotic dynamics of our quasiperiodic sums and products is asymptotically fractal (as is qualitatively demonstrated by the graphs of our examples). We use renormalisation to explore scale-invariant fine structure of these fractals. However, these techniques and results then apply to quasiperiodic problems arising within Quantum Field Theories and Statistical Physics.

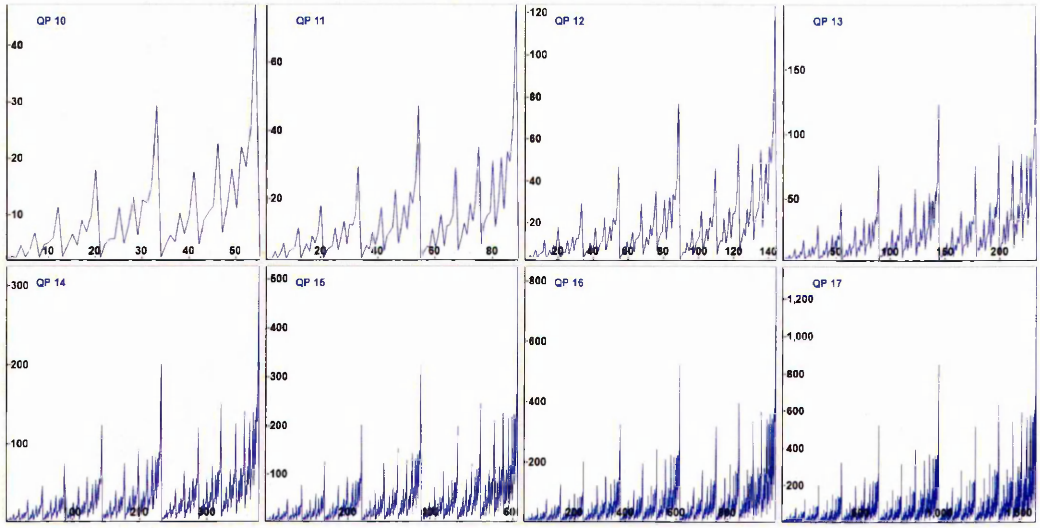

Figure 1.1.2: Graphs of the Sudler product of sines $P_n(\omega) = \prod_{k=1}^{n} 2\sin\pi\{k\omega\}$ against n for $n = 1 \dots F_m$ and $m = 10 \dots 17$. Each graph is the equivalent of the (rescaled) initial segment $[0, \omega]$ of its successor.

The particular renormalisation technique we will use differs from other techniques mainly in that the scaling factors to be used are dependent upon arithmetic properties of the factor α introduced in 1.1.1. We make use of the distinguished sequence of rationals (p_n/q_n) which is supplied by continued fraction theory and is convergent on α. The denominators (q_n) play a crucial role as described below. We will summarise the basic results we need from continued fraction theory in section 1.3.

The full process of renormalisation is normally carried out in two steps, namely DECIMATION and RESCALING. In the case of quasiperiodic renormalisation these steps are executed as follows:

1. Given a sequence (in n) of quasiperiodic sums $S_n(x, \alpha)$ we DECIMATE the sequence by extracting the subsequence $S'_n(x, \alpha) = S_{q_n}(x, \alpha)$.

2. The second step is carried can be carried out in two sub-steps as follows.

   (a) We fix x and introduce a new local variable y to focus on the immediate neighbourhood of x as follows: $S''_n(y) = S'_n(x + y, \alpha)$.

   (b) We introduce a RESCALING variable $y' = y/g(n)$ which depends on n, and allows us to define the RESCALED FUNCTIONS $S'''_n(y') = S''_n(y/g(n))$. We choose g to be a function converging to 0, so that if we keep y' constant, the equivalent value of y also converges to 0 with n. Hence the rescaled function $S'''_n(y')$ is describing the behaviour of $S_{q_n}(x, \alpha)$ over a smaller and smaller neighbourhood of x as n increases.

We do not always need to carry out the full process of renormalisation, and indeed we will not. In the early part of this work, we will only need to use the technique of decimation. We will

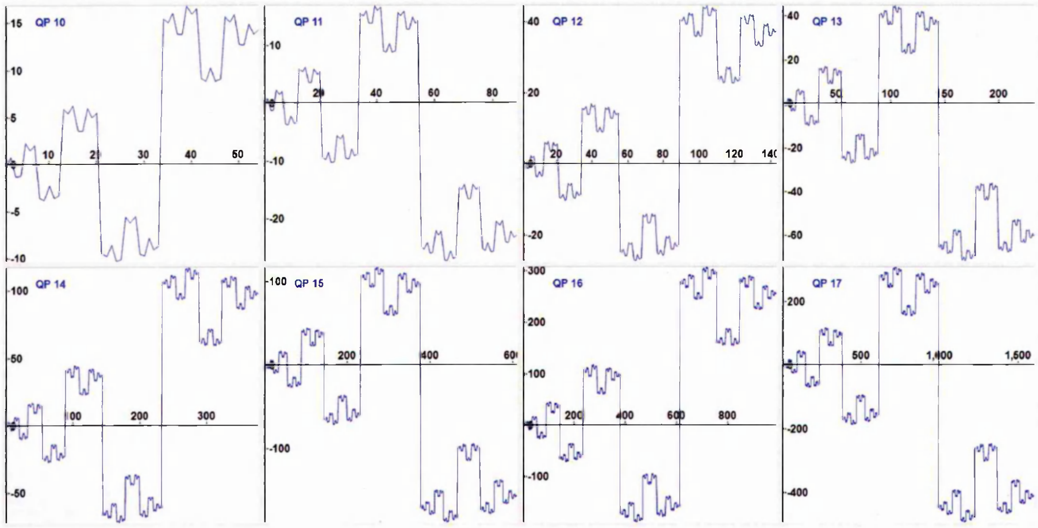

Figure 1.1.3: Graphs of the Knill sum of cotangents $S_n(\omega) = \sum_{k=1}^{n} \cot \pi\{k\omega\}$ against n for $n = 1 \dots F_m$ and $m = 10 \dots 17$. Each graph is the equivalent of inverting (and rescaling) the initial segment $[0, \omega]$ of its successor. This fascinating graph stabilises extremely quickly, and despite the unbounded nature of the value function f, the sum grows at a very controlled rate.

progress to use step 2(a) in later chapters, and we will only need to deploy the full process in the penultimate chapter.

1.1.4 Composition sum operators (CSOs)

We will define formally the CSOs we will work with in chapter 6, but provide a general introduction here. Suppose $\mathcal{A}$ and $\mathcal{F}$ are algebras of functions (ie closed under $+, \times$) such that $\mathcal{A}$ acts on $\mathcal{F}$ by composition (ie for any $\alpha \in \mathcal{A}, f \in \mathcal{F}$ we have $f \circ \alpha \in \mathcal{F}$). Then each α induces a natural COMPOSITION OPERATOR α^* on $\mathcal{F}$ defined by $\alpha^*(f) = f \circ \alpha$. These operators are linear and form their own algebra $\mathcal{A}^*$. There is also a larger algebra generated naturally by the direct product $\mathcal{F}\mathcal{A}^*$ of linear operators on $\mathcal{F}$ defined by $(f\alpha^*)g = f \times (g \circ \alpha)$. We call the general element $\sum_{i=1}^{n} f_i \alpha_i^*$ where $f_i \in \mathcal{F}$, $\alpha_i \in \mathcal{A}$ a COMPOSITION SUM OPERATOR (CSO).

CSOs arise naturally in a variety of situations. Again we give more detail in chapter 6, but give a few examples here.

1. CSOs arise in the study of the fixed points of quasiperiodic renormalisation. For example we will show (section 6.1) the fixed points of the golden renormalisation (ie the renormalisation group operator on quasiperiodic sums and products at the golden rotation) are the fixed points of the operator M defined by:

$$(Mf)(x) = f(-\omega x) + f(\omega^2 x + \omega) \tag{1.3}$$

Here $M = \alpha_1^* + \alpha_2^*$ is a CSO where $\alpha_1(x) = -\omega x$ and $\alpha_2(x) = \omega^2 x + \omega$. A large part of chapter 6 is devoted to finding the complete set of fixed points of this operator in a certain function space.

2. Ketoja and Satija[29] studied the quantum mechanical Harper equation (or almost Mathieu equation) and found self-similar fluctuations in the strong coupling limit. They conjectured the existence of a fixed point which, in the golden mean flux case, would satisfy a multiplicative version of the equation above, namely:

$$(M^{\times} f)(x) = f(-\omega x) f(\omega^2 x + \omega) \tag{1.4}$$

Here $M^{\times} = \alpha_1^* \times \alpha_2^*$ is a related CSO with α_1, α_2 as before. Mestel, Osbaldestin and Winn[46] later proved the existence of this particular fixed point and were able to construct it.

3. Gilbert[18, 19] studied kinematic dynamo theory and found a critical role played by spectral values of the operator T defined by:

$$(Tf)(x) = e^{i\alpha(x-1)/2} f\left(\frac{x-1}{2}\right) - e^{i\alpha(1-x)/2} f\left(\frac{1-x}{2}\right), \tag{1.5}$$

The operator T is a CSO of the form $f_1\alpha_1^* + f_2\alpha_2^*$ where $f_1(x) = e^{i\alpha(x-1)/2} = (-f_2(x))^{-1}$ and $\alpha_1(x) = (x-1)/2 = -\alpha_2(x)$. It has a spectral value of $\lambda \neq 0$ precisely when the CSO $\lambda^{-1}T$ has a fixed point.

1.2 Applications of quasiperiodic sums

In this section we summarise briefly some of the areas of mathematics where a unified study of quasiperiodic sums and products could make a significant contribution. We also outline how the work presented in this work extends previous work. A more detailed survey of existing results in the literature is provided in each subsequent chapter.

1.2.1 Diophantine approximation

Given an irrational α, the sequence often denoted $R(n\alpha)$ is the sequence of fractional parts $\{n\alpha\}$. It is of great importance within the related theories of Diophantine approximation, uniform distribution, and discrepancy analysis. In particular estimates of the size of the discrepancy sum $\sum_{k=1}^{n}(R(n\alpha)-\frac{1}{2})$ have been studied by Sierpinski, Hecke, Hardy & Littlewood, Ostrowski, Behnke, and Lang (see chapter 2 for details).

This sum is in fact the quasiperiodic sum $S_n(\alpha)$ introduced in section 1.1.1.1. We will show (chapter 2) that, by applying Denjoy-Koksma theory (part of ergodic theory) together with a technique of Ostrowski, we can sharpen previous bounds considerably and make good estimates of the constants involved.

1.2.2 Convergence rates of ergodic time averages

Ergodic theory is celebrated for establishing the theoretical foundation and pre-requisites for the methods of statistical physics. In particular it studies time averages, which, under a (measure preserving) map T, are the limiting values of arithmetic means of the form $\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$, or the analogous geometric means[2]. The theory is well matched to studying the physical systems for which it was developed, namely systems close to a long-term statistical equilibrium. However they are not well suited for studying systems in transition, or for determining how fast the computation of a time average will converge. These problems require understanding of the *rate* of convergence, and it is well known (eg [27]) that the ergodic theorems themselves can give little or no information about this.

However there are ways of linking T and f to give convergence estimates, and this problem has been studied intensively over the last 20 years by Kachurovskii and his school, using in particular the spectral measure of T with respect to f, the correlation coefficients $< f, fT^k >$, and the dispersion of f with respect to T (see [27] for the seminal paper).

The study of quasiperiodic sums and products provides a complementary approach. It shows that we can obtain good convergence rates by restricting f, T to more limited but nevertheless useful classes. For example, Birkhoff's (or von Neumann's) ergodic theorem (see eg [9]) tells us that the time average $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\{k\alpha\}$ has the value $\frac{1}{2}$ for almost all α. Our work in chapter 2 shows the rate of convergence to this limit of $\frac{1}{2}$ to be $O(\frac{1}{n}(\log n)^{2+\epsilon})$ for any $\epsilon > 0$ and for almost all α. More precisely $\left|\sum_{k=0}^{n-1}\left(\{k\alpha\} - \frac{1}{2}\right)\right| < C(\alpha) + \frac{3}{\log 2}(\log n)^{2+\epsilon}$ for some constant C depending only on α.

1.2.3 Critical KAM theory

Recall that the phase space of an integrable Hamiltonian system (eg the Newtonian model of a single planet Solar System) is typically foliated by invariant tori on which the orbits are typically quasiperiodic. Under perturbation (eg by the presence of another planet), many of these invariant tori are destroyed, leading to irregular, seemingly chaotic motion. The Kolmogorov-Arnol'd-Moser (KAM) theorem[35] famously established however that tori on which the quasiperiodic orbit has suitable arithmetic properties persist for small enough perturbations (thus allowing for the possibility of a stable multi-planet Solar System).

[2]The averages of the quasiperiodic sums and products we will study are just a special case of these, namely the case in which T is an irrational rotation of the circle.

Critical KAM theory studies the process of break-up of the tori. A key tool proposed by Horita et al (1989)[26] is the behaviour of the sum of local expansion rates along the orbit. This is closely related to the convergence rate of the Lyapunov time average (which still exists in these non-ergodic systems although it is typically not ae constant). More importantly for our purposes, in many well-studied standard models, this sum is a quasiperiodic sum. Most of the work to date however has been limited to numerical studies or heuristic arguments, and there is limited rigorous theory.

One key exception has been a series of papers by Knill[32, 33, 31], the first of which was co-authored with Tangerman and the second with Lesieutre. Knill and co-workers, in trying to develop a more easily computable approach to critical KAM theory, focused on developing a better understanding of the behaviour of the particular quasiperiodic sum:

$$S_n(\alpha) = \sum_{k=1}^{n} \log 2(1 - \cos 2\pi k\alpha)$$

Although they established other useful results along the way, their finding was that known results in ergodic theory failed to give optimal results (the same conclusion was reached by Lubinsky in [41], applying existing number-theoretic results). They developed a series of theoretical conjectures based on computer analysis. Our contribution will be to give the first complete and rigorous proof[3] of these results in chapter 4, and to extend them in chapter 5.

1.2.4 SNAs and nonlinear dynamical systems

Strange non-chaotic attractors (SNAs) arise in parametrised families of dissipative dynamical systems for parameter values which lie between ordered and chaotic regimes. They are therefore of great interest in studying the transition to chaos. *Non-chaotic* in this context means that the dynamics on the attractor have only non-positive Lyapunov exponents, and *strange* here means that the attractor has rough (non-differentiable, and often fractal) geometry. These attractors are easiest to observe in electronic systems (see eg [15]), but there are also papers reporting detection in areas as diverse as biological systems[50], glaciation[47], and star systems[40].

The majority of SNAs which have been studied to date arise as a result of quasiperiodic forcing, and most studies to date have been numerical. The main exception is that of quasiperiodically driven skew product maps on the cylinder. These are maps defined on the fibre product $\mathbb{T} \times \mathbb{R}$ in which the map on the fibre ($\mathbb{R}$) is driven by an irrational rotation of the base ($\mathbb{T}$), ie maps of the form $F : (x, y) \mapsto (\{x + \alpha\}, h(x, y))$. In particular we are interested in separable functions h so

[3] A reviewer recently brought to our attention Knill's latest paper on the subject (which we had missed as it has only appeared on the arxiv to date). Here Knill sketches an outline proof based on a different approach. Our experience of this area is that there could be some problems in carrying through this approach. However if it can be carried through it will provide a very elegant solution.

that the map hs the form $F : (x, y) \mapsto (\{x + \alpha\},\ f(x)g(y)\,)$.

In these systems g plays the role of a governing function ensuring that the y-coordinate remains bounded. Examples which have been studied are $g(y) = \tanh(y)$ [20] or $g(y) = y/\sqrt{1+y^2}$ [37]. Given an initial condition (x_0, y_0) it is easy to show that the nth iterate $F^n(x_0, y_0)$ has a y-coordinate of the form $P_n(x_0)Q_n(x_0, y_0)$, where $|Q_n(x_0, y_0)|$ is a sequence of values which are monotonic decreasing with n from $|y_0|$, and where P_n is the quasiperiodic product:

$$P_n(x_0) = \prod_{k=0}^{n-1} f(x_0 + k\alpha)$$

This means that the interesting behaviour of these systems is driven by the growth of the quasiperiodic product $P_n(x)$. Our new results on $P_n(\omega)$ in this work give improved understanding of the nature and development of these SNAs in the critical neighbourhood around the x-axis.

1.2.5 q-series

The theory of q-series and q-analogues is an area of pure mathematics which extends the work on hypergeometric series carried out by Euler, Cauchy, Gauss, Riemann, Ramanujan and many others (see eg [12]). However it has also found great application in theoretical physics, particularly string theory, quantum groups and superalgebras (see eg [56, 1]). A central piece of machinery is the q-Pochhammer symbol $(a; q)_n$ where $a, q \in \mathbb{C}$ and $n \in \mathbb{Z}$. In recent years there has been increasing interest in the growth with n of the q-Pochhammer symbol in the special case $a = q \to 1, n > 0$. Lubinsky [41] lists 13 recent papers in which this question has begun to figure prominently. When we put $n > 0$ and $a = q$ and then let $|q| \to 1$, the value of $(a; q)_n$ becomes equal to $P_n(\alpha)$ where $\alpha = \arg(q)$. Our results in chapter 5 improve on the best known results for the golden rotation case $\alpha = \omega = \left\{\frac{1}{2}(\sqrt{5}+1)\right\}$, in the process settling an open question of Erdős-Szekeres-Lubinsky dating back to 1959 [11, 41].

1.2.6 Partition theory

Euler first exhibited the remarkable connection between the Euler function $E(x) = \prod_{k=1}^{\infty}(1 - x^k)$ and partitions of integers. By taking the power series expansion $(E(x))^{-1} = \sum_{k=0}^{\infty} a_k x^k$ and examining the construction of the coefficients, it is easy to show that a_n is the number of different sets of (strictly) positive integers whose sum is n (each set is a *partition* of n). And, if we write $E(x) = \sum_{k=0}^{\infty} b_k x^k$, then b_n is the number of even partitions of n (sets with an even number of positive integers summing to n) less the number of odd partitions. Euler went on to prove the celebrated identity $E(x) = \sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n-1)/2}$ (the pentagonal number theorem - see [6] for a comprehensive review).

This leads to consideration of the the finite polynomials $E_n(x) = \prod_{k=1}^{n}(1-x^k)$. If $(E_n(x))^{-1} = \sum_{k=0}^{\infty} a_{nk}x^k$ we now find that a_{nk} is the number of partitions of k into sets of integers summing to k, but which are this time restricted to a maximum value of n. And if $E_n(x) = \sum_{k=0}^{\infty} b_{nk}x^k$ then b_{nk} represents the number of even partitions less the odd partitions of k, again restricted to a maximum value of n in each partition.

We call $E_n(x)$ a restricted Euler function, and it is natural to ask whether there is a restricted version of the pentagonal number theorem. This time a closed form formula is not known, and instead research has focused on the growth rate of the b_{nk}. Remarkably (see [55]) it turns out that for any given n, $\frac{1}{n^2}\,\|P_n(\alpha)\| \leq \max_k\{|b_{nk}|\} \leq \|P_n(\alpha)\|$ using the supremum norm over α. Hence the growth rate of $|b_{nk}|$ is closely linked to the norm of the Sudler product. Researchers have developed increasingly refined estimates of this norm over the years, including both first and second order terms, but fundamentally the growth of the norm is exponential (see section 4.1.1 for more details).

Lubinsky[41] studied the quasiperiodic product itself, ie $P_n(\alpha)$ rather than its norm, and discovered the surprising result that this grows almost everywhere at power law rates, not exponential rates. Our results (chapter 5) improve this further to linear growth when α is the golden ratio, so that the norm growth is in fact wildly exaggerated in comparison with typical growth. Our analysis (section 4.1.1) also suggests that the growth is eventually limited to power law *everywhere* (ie there are no points at all which have exponential growth). The exponential norm growth is instead caused by the fact that the growth at each value α undergoes a period of exponential inflation over a limited range of n, but then reverts to power law growth.

1.2.7 Complex analysis and dynamics

If we take the Euler functions discussed in the previous section and regard them as complex functions, then $E(z) = \prod_{k=1}^{\infty}(1 - z^k)$ is a power series around 0 with radius of convergence 1. However its convergence on the unit circle is much less clear. The zeroes of the power series $E(z)$ are dense on this circle (corresponding to the points of the circle whose arguments are a rational multiple of π). Values at non-rational arguments are less easy to compute, so a natural approach is to study the restricted Euler functions E_n. Knill [33] points out that this is equivalent to studying the complex dynamical system on $\mathbb{C}^2$ given by $T : (z, w) \mapsto (cz,\ w(1 - z)\,)$ for some $|c| = 1$.

It is easy to show that if $z = e^{2i\pi\alpha}$ then $|E_n(z)| = P_n(\alpha)$. Since we know (Lubinsky [41]]) that $P_n(\alpha)$ is unbounded for irrational α, it follows that $E(z)$ is zero for rational α and undefined everywhere else. Our own results in chapter 4 show however that the sequence $E_n(z)$ does not simply diverge with n: there are values of z for which the sequence has non-trivial (ie other than $\{0, \infty\}$) points of accumulation, and in particular the sequence has a convergent subsequence defined by its quasiperiodic decimation. We can further say that the complex dynamical system of Knill has orbits which wander further and further from the origin in each quasiperiod, but return ever closer to a non-trivial ω-limit point at the end of each quasiperiod.

1.2.8 Summary

Quasiperiodic sums and products have been studied in a wide variety of mathematical disciplines, but are often given different names and notations which obscures the underlying commonality. However they seem to be playing an increasingly important role in various areas of current mathematical focus, and there is a good case for unifying their study. By bringing together techniques of number theory, ergodic theory, and renormalisation we will in show in succeeding chapters that we can significantly improve a number of classical results, as well as developing new ones.

1.3 Further notation and preliminaries

We introduced our central notation for quasiperiodic sums and products in section 1.1. Here we will add some details and recall some elementary theory needed in subsequent chapters.

1.3.1 Quasiperiodic sums

We will use the notation $||.|| : \mathbb{T} \to [0, 1/2]$ to denote the function giving the shortest distance of a point from the point 0 in either direction (eg $||3/4|| = 1/4$). Its extension $||.|| : \mathbb{R} \to [0, 1/2]$ gives the distance to the nearest integer.

Given a quasiperiodic product $P_n(x, \alpha) = \prod_{k=0}^{n-1} f(x + k\alpha)$ we will adopt the terminology of dynamical systems and refer to x as the INITIAL CONDITION, and α as the ROTATION NUMBER.

In general we can allow the value function f to assign values in general spaces. In particular when the range of f is a space of matrices, the quasiperiodic products[4] of f becomes a cocycle (and provides perhaps a rather more motivational approach to cocycles than do some of the formal definitions). However in this work we will only make use of the extended real line as the range for value functions.

When $f \geq 0$ (ie $f(x) \geq 0$ for all x), we can derive a quasiperiodic sum from each quasiperiodic product by taking logarithms, ie $S_n(x, \alpha, \log f) = \log P_n(x, \alpha, f) = \sum_{k=0}^{n-1} \log f(x + k\alpha)$ (here we define $\log(0) = -\infty$). We call this sum the ADDITIVE FORM of $P_n(x, \alpha, f)$.

[4]There are of course two products due to the non-commutative nature of matrix multiplication

1.3.2 Continued fractions

We will need some elementary results from continued fraction theory (see eg [21] for derivations). Recall in particular, that given $\alpha \in [0, 1)$ there is a distinguished sequence[5] (p_n/q_n) of positive rationals in lowest form which converges to α (the CONVERGENTS of α). The convergence is optimal in the sense that for $q < q_n$ we have $|q_n\alpha - p_n| < |q\alpha - p|$. Defining $\alpha_n = |q_n\alpha - p_n|$ we have $\alpha_n < 1/q_{n+1}$ and the convergence is alternating, ie

$$q_n\alpha - p_n = (-1)^n\alpha_n \tag{1.6}$$

We will make use of the identity

$$p_{n+1}q_n - p_nq_{n+1} = (-1)^n \tag{1.7}$$

The integers $a_n = \lfloor q_{n+1}/q_n \rfloor \geq 1$ for $n \geq 0$ are the PARTIAL QUOTIENTS of α and have the property that the sequences (p_n) and (q_n) both satisfy the recurrence:

$$t_{n+1} = a_nt_n + t_{n-1} \tag{1.8}$$

with initial values $(p_0, p_1) = (q_{-1}, q_0) = (0, 1)$. In particular this gives us $q_1 \geq 1$, $q_n > q_m$ for $n > m \geq 1$ and $q_{n+1} \geq 2q_{n-1}$ for $n \geq 1$, and hence for $n \geq 2$

$$q_n \geq 2^{n/2} \tag{1.9}$$

We combine this with a dynamical systems perspective. We note that:

$$\|\{q_n\alpha\}\| = \|\{q_n\alpha - p_n\}\| = \alpha_n < 1/q_{n+1} \tag{1.10}$$

so that the sequence $(\{q_n\alpha\})$ converges to 0 on the circle. We will refer to q_n as a QUASIPERIOD - it is the time at which the orbit of x returns closest to its starting point. For a rational rotation number $\alpha = p/q$, the final convergent to α is precisely p/q and then the quasiperiod coincides with the period.

1.3.3 Diophantine approximation

Finally we will reference two results of Diophantine approximation.

First we say α is of *constant type c* if there is a constant $c > 0$ such that for all n we have

[5]The sequence is uniquely determined and infinite for irrational α. In the case of rational α there are two possible sequences which are finite and differ in length by 1. We take the shorter of these as the distinguished sequence.

$\alpha_n = \|q_n\alpha\| > 1/(cq_n)$. It is well-known that all quadratic irrationals are of constant type (see eg Hardy & Wright [21]), and in particular this includes the golden rotation $\omega = \left\{\frac{1}{2}\left(\sqrt{5}+1\right)\right\}$.

Secondly Khintchine [30] showed that if a series $\sum_0^\infty \Psi_n$ of positive numbers converges, then for (Lebesgue) almost all α we have $\alpha_n = \|q_n\alpha\| \geq \Psi_{q_n}$ for all large enough n. If for any $\epsilon > 0$, α satisfies $\alpha_n = \|q_n\alpha\| > 1/q_n(\log q_n)^{1+\epsilon}$ for large enough n, we say α is of *supra-log type*. Then Khintchine's result tells us that the numbers of supra-log type have full measure.

We will use these two classes of irrationals (those of constant type, and those of supra-log type) in chapters 2 and 3.

1.4 Overview of this document

1.4.1 Overview of the structure

In chapters 2-5 we will study the three quasiperiodic sums introduced in section 1.1. Chapter 2 is devoted to the sum of remainders, and chapter 3 to the Knill sum of cotangents. We study these first as we will need the results of both in our study of the Sudler product of sines. The study of the Sudler product seems to require rather more work, and we have split it into two chapters. In chapter 4 we focus on providing a rigorous proof of some recently reported experimental results ([33]), whilst in chapter 5 we extend our proof to obtain completely new results. In this set of chapters we will make increasing use of quasiperiodic renormalisation as a technique.

In chapter 6 we switch to studying fixed points of the quasiperiodic renormalisation group, and this leads us to study fixed points of the more general class of linear operators which we call composition sum operators.

In chapter 7 we conclude that we have taken just a first few steps in both of these areas, and map out a goodly number of topics which seem to suggest themselves for further research.

1.4.2 Overview of the main results

This section is meant as a convenient summary - the reader is directed to the appropriate chapter for the full definitions of these results.

Chapter 2. We study the sum of remainders $S_n(\alpha) = \sum_{r=1}^n \{r\alpha\} - 1/2$. We will build on the results of Hecke, Hardy & Littlewood, Lang et al, namely that for almost all α, $S_n(\alpha) = O\left((\log n)^{2+\epsilon}\right)$ for any $\epsilon > 0$. We improve the result to $S_n(x,\alpha) = O\left((\log n)^{2+\epsilon}\right)$, ie it holds for any x and not just $x = \alpha$. We also show this is a poor estimate for most values of n, and that a much sharper estimate is given by

$$\left|S_n^{Rem}(x)\right| \leq \frac{3}{2}\sum b_i \tag{1.11}$$

where $\sum b_i q_i$ is the Ostrowski representation of $n \geq 0$ (see 2.5.1), and the numbers q_i are quasiperiods of α. In particular this shows that for an infinite set of $n \in \mathbb{N}$, we have $|S_n(x, \alpha)| \leq \frac{3}{2} = O(1)$.

Chapter 3. We develop bounds for the Knill sum of cotangents over a quasiperiod:

$$\left|S_{q_n}(\alpha)\right| = \left|\sum_{r=1}^{q_n} \cot \pi r \alpha\right| < \frac{1}{\pi \alpha_n} + \frac{\pi \alpha_n}{2} \tag{1.12}$$

In particular at the golden ratio $\alpha = \omega$ this gives us:

$$|S_{F_n}(\omega)| = \left|\sum_{r=1}^{F_n} \cot \pi r \omega\right| \quad < \quad \frac{1}{\pi \omega^n} + \frac{\pi \omega^n}{2} \tag{1.13}$$

which in turn means that growth is at most linear.

Chapter 4. We prove a foundational result for the Sudler product of sines at the golden ratio and over a quasiperiod, namely that as $n \to \infty$, there is a constant c such that:

$$P_{F_n}(\omega) = \prod_{r=1}^{F_n} |2 \sin \pi r \omega| \to c \tag{1.14}$$

We use this result to give a new proof of the fact that this product has power law growth.

Chapter 5. We extend the methods of the previous chapter to prove new results, namely that there is a constant P such that for each $F_n \leq k < F_{n+1}$

$$1 < P_{F_n}(\omega) \leq P_k(\omega) \leq Pk \tag{1.15}$$

Hence the product is bounded by linear growth above, and a constant below. In particular this settles (negatively) an open question of Erdős-Szekeres-Lubinsky dating initially from 1959.

Chapter 6. We develop some basic theory of unbounded (non-Banach) vector spaces. In particular, given a linear operator T on a general vector space, we show that under certain quite general conditions we can construct a derived operator $\hat{T}$ whose image is the subspace of fixed points of T.

When T is a composition sum operator we develop conditions under which T may have fixed points with PESL singularities, and the necessary structure of these fixed points.

For the golden renormalisation operator M (which is a composition sum operator) we use theory to find the fixed points of M with PESL singularities, and show that this set of fixed points is complete.

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Chapter 2

The sum of remainders

We start with probably the simplest and best understood quasiperiodic sum, which we introduced in section 1.1.1.1. It is defined using the value function $f(x) = x - \frac{1}{2}$ to construct the sum $S_n(x, \alpha) = \sum_{k=0}^{n-1} \left(\{x + k\alpha\} - \frac{1}{2}\right)$. The main result of this chapter is:

Theorem. *For irrational α, and any real x, $|S_n(x, \alpha)| \leq C_\alpha O_\alpha(n)$ where $C_\alpha = \sup_n \left\{\left|S_{q_n}(x, \alpha)\right|\right\} \leq 3/2$, and $O_\alpha(n)$ is the Ostrowski digit sum of n (see 2.5.2)*

In particular there are an infinite number of integers n satisfying $|S_n(x, \alpha)| \leq \frac{3}{2}$.

2.1 Existing results

The base case $S_n(\alpha) = S_n(\alpha, \alpha)$ has been studied intensively. Sierpinski (1909) first obtained the result that for any irrational α, $S_n(\alpha) = o(n)$. Improved results were obtained for particular classes of α by other authors including Hecke[24], Hardy & Littlewood[22], Ostrowski[49], and Behnke[4].

An important theoretical contribution was made a little later by Khintchine (1926)[30] who found a unifying and simplifying classification scheme for the amalgam of classes of α previously studied, using a type function. This was refined by Lang (1966)[38] and allowed him to unify the previous results in one very elegant result as follows:

Definition 2.1.1 (Lang). Let $f : [1, \infty) \to [1, \infty)$ be an increasing function. We say $\alpha \in \mathbb{R}$ is of Lang type f[1] if for all sufficiently large integers B there exists a solution in relatively prime integers p, q of the inequalities:

$$|q\alpha - p| < 1/q \text{ and } B/f(B) \leq q < B$$

[1]Lang's precise notation was to write "$\leq f$" but we will simply write "f".

Theorem 2.1.2 (Lang). *Let α be of Lang type f and assume $f(t)/t$ is (monotone) decreasing. Then*

$$S_n(\alpha) = O\left(\int_1^n \frac{f(t)}{t} dt\right)$$

In particular, when α is of type $\lambda(\log t)^\gamma$ for some $\gamma \geq 0$ then $S_n(\alpha) = O\left((\log n)^{1+\gamma}\right)$.

2.2 Overview of new results

Theorem 2.1.2 is powerful, and within its original theoretical purpose it is perhaps optimal. However, for our slightly more practical purposes there are some important improvements we proceed to develop in this chapter.

Firstly, Lang's definition of type leads to elegant results, but it is not the most transparent. We will introduce a new definition which is clearer and which suits our purposes better. It is slightly stricter than Lang's definition, in the sense that results which hold with our definition will also hold with Lang's.

Secondly, the rise of the digital computer has focused attention in number theory on "effective" results, meaning results which are both computable and useful (ie not extravagantly far from actuals - see eg Baker's Fields Medallist lecture (1997)[2]). We will improve on the "big O" notation of theorem 2.1.2 by deriving actual and useful constants for the growth rate.

Finally, Lang's result is very much a bound on the peaks of the sequence $|S_n(\alpha)|$ as it is strictly increasing with n, and hence never less than the largest preceding peak. However we see from the graph of S_n in section 1.1.1.1 that for many values of n, the value $|S_n(\alpha)|$ is very much lower than the preceding peak, so that this bound is sub-optimal at non-peak points in the sequence. We will provide a pointwise bound which is much sharper at non-maximal points.

2.3 A new definition of Diophantine type

Recall (section 1.3) that for any given real number α, the theory of continued fractions provides us with a sequence (p_n/q_n) of rationals (in lowest form) which are best rational approximations to α. Much of our work will be in terms of the quasiperiods (q_n). We therefore give a definition in terms of these quasiperiods.

Definition 2.3.1 (Quasiperiodic type). Let $f : [1,\infty) \to [1,\infty)$ be increasing, and let (q_n) be the sequence of quasiperiods of a positive real α. Then α is of quasiperiodic type f if there is an $N \geq 0$ such that $q_{n+1}/q_n \leq f(q_n)$ for all $n \geq N$.

Lemma 2.3.2. *If α is of quasiperiodic type f, it is also of Lang type f.*

Proof. We show that we can satisfy the two Lang requirements.

By equation (1.6), for each convergent we have $|q_n\alpha - p_n| < 1/q_{n+1}$ and since $q_{n+1} > q_n$, every p_n/q_n satisfies the first requirement of definition 2.1.1, that $|q\alpha - p| < 1/q$.

We now show that the second condition is also satisfied. For $n \geq N$ we have by definition $q_{n+1}/f(q_n) \leq q_n$. So for any $q_n < B \leq q_{n+1}$ we have $B/f(q_n) \leq q_{n+1}/f(q_n) \leq q_n. < B$. But given any $B > q_0 = 1$ we can always find n such that $q_n < B \leq q_{n+1}$ and so the second condition is satisfied for any $B > q_N$. □

2.3.1 Constant type and supra-log type

We now relate the quasiperiodic type of α to the two classes introduced in section 1.3.3.

First, if α is of constant type in the sense of section 1.3.3, there is a $c > 0$ with $\alpha_n = \|q_n\alpha\| \geq 1/cq_n$. But by (1.6) $\|q_n\alpha\| < 1/q_{n+1}$ and hence $q_{n+1}/q_n \leq c$, and so α is of constant quasiperiodic type ($f(t) = c$), and hence also of constant Lang type.

Second, from the same section, α is of supra-log type for almost all α, and then $\|q_n\alpha\| \geq 1/q_n(\log q_n)^{1+\epsilon}$ for any $\epsilon > 0$ and all large enough n. But by (1.6) $\|q_n\alpha\| < 1/q_{n+1}$ and this gives us for large enough n that $1/q_{n+1} > 1/q_n(\log q_n)^{1+\epsilon}$, or $q_{n+1}/q_n < (\log q_n)^{1+\epsilon}$. Hence almost all α are also of quasiperiodic type $f(n) = (\log n)^{1+\epsilon}$, and hence of Lang type $(\log n)^{1+\epsilon}$.

2.4 Constants of the growth rate

Our goal in this section is to study the bounds of $S_n(x,\alpha)$ over a quasiperiod. We introduce some results from the ergodic theory of the circle. We first recall a foundational result of Poincaré.

Let T be an orientation preserving homeomorphism of the circle $\mathbb{T} = [0,1)$. We define the value function $f(x) = \{Tx - x\}$ and note that $f(x)$ gives the forwards rotation at x, ie the distance by which T moves x measured *forwards* along the circle $[0,1)$. We can now define the sum $S_n(x,T) = \sum_{k=0}^{n-1} f(T^k x) = \sum_{k=0}^{n-1}\left\{T^{k+1}x - T^k x\right\}$), noting that this is a simple generalisation of a quasiperiodic sum, and indeed is precisely a quasiperiodic sum when T is a rigid rotation. $S_n(x,T)$ also represents the cumulative forwards rotation of $x$under T, and so the average rotation of T along the orbit of x (if it exists) is given by the path average of f, namely $\lim_{n\to\infty}\frac{1}{n}S_n(x,T)$.

Theorem 2.4.1 (Poincaré). *Let T be an orientation preserving homeomorphism of the circle $\mathbb{T} = [0,1)$. Then the limit of the average rotation*

$$\rho_T(x) = \lim_{n\to\infty}\frac{1}{n}S_n(x,T)$$

along the path of any $x \in \mathbb{T}$, exists, and further $\rho_T(x)$ is a constant function.

A proof can be found in most textbooks on dynamical systems (see eg Katok & Hasselblatt[28]). The constant $\rho_T = \rho_T(x)$ is called the ROTATION NUMBER of T. Note that for a rigid rotation by α, the value function is simply $f(x) = \{Tx - x\} = \{\{x+\alpha\} - x\} = \alpha$, and so the rotation number is $\lim \frac{1}{n}(n\alpha) = \alpha$.

Remark. This result is usually presented in terms of lifts of functions, or representation functions, but this approach using Birkhoff sums provides an alternative presentation and further clarifies this result as an early ergodic theorem.

We now recall a powerful result concerning functions on the circle called the Denjoy-Koksma inequality. It was introduced by Herman[25](1979) who strengthened a lemma of Denjoy using an approach from Koksma's proof of the Koksma-Hlawka inequality (an important result in discrepancy theory). However Herman's monograph was in French. When Cornfeld, Fomin and Sinai's influential textbook "Ergodic Theory" was translated into English in 1982[9], it introduced a variant of the result which is often cited. However some care is required here, as the latter result is given for continuous functions only. We need a result we can use with the value function $f(x) = x - \frac{1}{2}$, and this is discontinuous (on the circle) at the point 0, where it jumps from $+\frac{1}{2}$ to $-\frac{1}{2}$. Fortunately Herman's version does not require continuity (he even states this explicitly in his statement of the theorem), it only requires that the function be of bounded variation. (Herman's statement does fail to mention explicitly the necessary condition that the function be integrable (it is just assumed), and it is possible that this is why continuity was added to the textbook version.)

We will adopt the Katok-Hasselblatt[28] definition of bounded variation, as it slightly simplifies the proof over using the usual (partition based) definition.

Definition 2.4.2. Let J be a finite set of intervals $\{[x_k, y_k]\}$ of $\mathbb{T}$ which are disjoint with the possible exception of endpoints. A function $f : \mathbb{T} \to \mathbb{R}$ is of bounded variation if $\sup\left(\sum_k |f(x_k) - f(y_k)|\right) < \infty$ where the supremum is taken over all possible sets J. The supremum value is called the total variation, denoted $\mathrm{Var}(f)$.

Note that here J may be a partitioning of the circle, but it is not required to be.

We also note that $f(x) = x - \frac{1}{2}$ is of bounded variation on $\mathbb{T} = [0, 1)$ with total variation 2, as it rises continuously from $-1/2$ to $(1/2)^-$ and then drops back to $-1/2$.

Theorem 2.4.3 (Herman). *Let μ be a measure on the circle, and T an orientation and measure preserving homeomorphism of the circle with irrational rotation number α. Suppose p/q is in lowest terms and satisfies $|q\alpha - p| \leq 1/q$. Finally let $f : \mathbb{T} \to \mathbb{R}$ be μ-integrable and of bounded variation with path sum $S_n(x, T, f) = \sum_{k=0}^{n-1} f(T^k x)$. Then for any $x \in \mathbb{T}$ we have*

$$\left| S_q(x, T, f) - q \oint f \, d\mu \right| \leq \mathrm{Var}(f)$$

We can apply this theorem to the remainder sum, noting that α must be irrational. Note that any rotation is orientation preserving, and also preserves Lebesgue measure. The value function $f(x) = x - \frac{1}{2}$ is Lebesgue integrable on the circle with $\oint f d\mu = 0$, and is of bounded variation $\mathrm{Var}(f) = 2$. Finally as noted above any convergent p_n/q_n of α satisfies the $|q\alpha - p| \leq 1/q$. The theorem is therefore applicable and gives us $|S_{q_n}(x)| \leq 2$.

At this point we only have a bound for a subsequence of values (namely at the points (q_n)), but we shall see that this can be extended to provide a bound everywhere. We first conclude this section by obtaining a further improvement of the bound for the values at (q_n).

The power of the Denjoy-Koksma inequality has given us the bound of 2 for very little effort. However it uses the overall variation in any sub-interval as the primary estimating tool. By modifying the standard proof, we can substitute improved estimates based on the linearity of the value function $f(x) = x - \frac{1}{2}$. This results in an improved bound as follows:

Lemma 2.4.4. *For any real α let p/q (in lowest terms) satisfy $|q\alpha - p| < 1/q$. Then for any real x*

$$\left|S_q(x, \alpha)\right| < \frac{3}{2}$$

Proof. We can assume without loss of generality that $\alpha, x \in [0, 1)$. The condition on p/q gives us $\alpha - p/q = \nu/q^2$ for some $|\nu| < 1$.

Now $x = k/q + \phi$ for some $0 \leq k < q$ and $0 \leq \phi < 1/q$, and so $x + i\alpha = (k + ip)/q + \phi + i\nu/q^2$.

Suppose $\nu \geq 0$, then for $1 \leq i \leq q$ we have $(k + ip)/q \leq x + i\alpha < (k + ip + 2)/q$ and so $\{(k + ip)/q\} \leq \{x + i\alpha\}$ with the one exception that when $k + ip \equiv -1 \bmod q$ we may have $\phi + i\nu/q^2 \geq 1/q$ and then we can only write $\{(k + ip)/q\} - (q - 1)/q \leq \{x + i\alpha\}$.

Now $(p, q) = 1$, and so as i runs through $1, \ldots, q$, $k + ip$ runs through a complete set of residues $0, \ldots, q - 1 \bmod q$, and hence

$$\sum_{i=1}^{q}\{x + i\alpha\} \geq \left(\sum_{j=0}^{q-1}\frac{j}{q}\right) - \frac{q-1}{q} = \frac{1}{2}(q - 1) - \frac{q-1}{q} \tag{2.1}$$

Similarly for $\nu < 0$ we have $(k+ip-1)/q < x+i\alpha < (k+ip+1)/q$ and so $\{(k + ip - 1)/q\} < \{x + i\alpha\}$ with the one exception that when $k + ip - 1 \equiv -1 \bmod q$ we may have $\phi + i\nu/q^2 \geq 0$ and then we can only write $\{(k + ip - 1)/q\} - (q - 1)/q \leq \{x + i\alpha\}$. Now summing as before also gives (2.1), and so this holds for any $|\nu| < 1$. We can immediately deduce

$$\sum_{i=1}^{q}\left(\{x + i\alpha\} - \frac{1}{2}\right) > -\frac{3}{2} \tag{2.2}$$

We now examine the upper bound of the sum. For $\nu \geq 0$ we have

$$\sum_{i=1}^{q}\{x+i\alpha\} \leq \sum_{j=0}^{q-1}\frac{j}{q}+\sum_{i=1}^{q}\left(\phi+\frac{i\nu}{q^2}\right) = \frac{1}{2}(q-1)+q\phi+\frac{1}{2}(q+1)\frac{\nu}{q} < \frac{1}{2}(q-1)+1+\frac{1}{2}(1+\frac{1}{q}) = \frac{1}{2}q+1+\frac{1}{2q} \tag{2.3}$$

whilst for $\nu < 0$ we get

$$\sum_{i=1}^{q}\{x+i\alpha\} \leq \sum_{j=1}^{q}\frac{j}{q}+\sum_{i=1}^{q}\left(\frac{i\nu}{q^2}\right) = \frac{1}{2}(q+1)+\frac{1}{2}(q+1)\frac{\nu}{q} < \frac{1}{2}(q+1) \tag{2.4}$$

By adding $\sum_{i=1}^{q}(-1/2) = -(1/2)q$ to the two upper bound inequalities, the result follows by combining the three bounds obtained. □

Corollary 2.4.5. *Since any convergent p_n/q_n of α satisfies the condition on p/q, we deduce $|S_{q_n}(x,\alpha)| < 3/2$.*

Remark. This result does not require α to be irrational. However if α is irrational the sequence (q_n) provides an infinite set of k for which $S_k(x,\alpha)$ is bounded.

2.5 Pointwise bounds

We noted that Lang's result gives bounds for the peaks of the sequence $|S_n(\alpha)|$, but that $S_n(\alpha)$ varies energetically with n. We would like to obtain rather better bounds for values of n at which $S_n(\alpha)$ is far from the preceding peak. To study these values we will use a technique introduced by Ostrowski (1922)[49] (and rediscovered by Zeckendorf(1972)[60] some years later in the special case of the golden ratio).

2.5.1 Ostrowski representation

Let (q_n) be the sequence of quasiperiods of an irrational α, and recall that $q_0 = 1$. Ostrowski's result is that we can represent any $N \geq 0$ as a sum $\sum_{i=0}^{n} b_i q_i$ in a canonical manner, as follows:

Definition 2.5.1. Ostrowski representation. Represent $N = 0$ by the empty sum. Note from section 1.3.2 that (q_n) is an increasing sequence (strictly increasing for $n \geq 1$). Hence for $N > 0$ we can always find the maximum n with $q_n \leq N < q_{n+1}$. Put $1 \leq b_n = \lfloor N/q_n \rfloor$ and then $N = b_n q_n + (N - b_n q_n)$ with $0 \leq N - b_n q_n < q_n$. Repeat the algorithm with $N - b_n q_n$ until the remainder is 0.

Note that the approach just described defines a recursive algorithm in N which is computable in order $\log N$ steps (given q_n). The representation also allows us to define the following function which will play a central role in our study of pointwise bounds:

Definition 2.5.2. Given an irrational α, the OSTROWSKI DIGIT SUM of $N \geq 0$ with respect to α is the sum $O_\alpha(N) = \sum_{i=0}^{n} b_i$ of the coefficients of the Ostrowski representation $N = \sum_{i=0}^{n} b_i q_i$.

2.5.2 Bounds

We will now use the results of the previous section to establish our main result. First we establish a simple general lemma (recalling that the empty sum $S_0(x) = 0$):

Lemma 2.5.3. *For $n \geq 0$, let $S_n(x, \alpha)$ be a quasiperiodic sum over a rotation α. Then for any $0 \leq m \leq n$ we have $S_n(x, \alpha) = S_m(x, \alpha) + S_{n-m}(x + m\alpha, \alpha)$*

Proof. $S_n(x, \alpha) = \sum_{k=0}^{n-1} f(x + k\alpha) = \sum_{k=0}^{m-1} f(x + k\alpha) + \sum_{k=m}^{n-1} f(x + k\alpha) = \sum_{k=0}^{m-1} f(x + k\alpha) + \sum_{k=0}^{n-1-m} f(x + m\alpha + k\alpha) = S_m(x, \alpha) + S_{n-m}(x + m\alpha, \alpha)$ □

Remark. We have proved this result in additive notation, but analogously we have $P_n(x, \alpha) = P_m(x, \alpha) P_{n-m}(x + m\alpha, \alpha)$.

Theorem 2.5.4. *For irrational α, and any real x, $C_\alpha = \sup_n \left\{ \left| S_{q_n}(x, \alpha) \right| \right\} \leq 3/2$, and $|S_n(x, \alpha)| \leq C_\alpha O_\alpha(n)$ where $O_\alpha(n)$ is the Ostrowski digit sum of n*

Proof. We know from corollary 2.4.5 that for irrational α, $\left| S_{q_n}(x, \alpha) \right| < 3/2$ and hence C_α exists and is $\leq 3/2$. Now recall $S_{N+1}(x, \alpha) = \sum_{k=0}^{N} \left(\{x + k\alpha\} - \frac{1}{2} \right)$. Determining p from the Ostrowski representation $N + 1 = \sum_{i=0}^{p} b_i q_i$, we can put $m = N + 1 - q_p$ and use lemma 2.5.3 to obtain:

$$
\begin{aligned}
|S_{N+1}(x, \alpha)| &= \left| S_{N+1-q_p}(x, \alpha) + S_{q_p}(x + (N + 1 - q_p)\alpha, \alpha) \right| \\
&\leq \left| S_{N+1-q_p}(x, \alpha) \right| + C_\alpha
\end{aligned}
$$

We now adopt the inductive hypothesis $|S_n(x, \alpha)| \leq C_\alpha O_\alpha(n)$ for $n \leq N$, and note it is true for $n = 0$ (the empty sum). This gives us:

$$
|S_{N+1}(x, \alpha)| \leq C_\alpha O_\alpha(N + 1 - q_p) + C_\alpha \tag{2.5}
$$

But $O(N + 1 - q_p) = O\left(\left(\sum_{i=0}^{p} b_i q_i \right) - q_p \right) = \left(\sum_{i=0}^{p} b_i \right) - 1 = O_\alpha(N + 1) - 1$. We use this in (2.5) to obtain

$$
|S_{N+1}(x, \alpha)| \leq C_\alpha (O(N + 1) - 1) + C_\alpha = C_\alpha O_\alpha(N + 1)
$$

which establishes the induction. □

Corollary 2.5.5. *Since $C_\alpha \leq 3/2$ we have proved $|S_N(\alpha)| = \sum_{k=0}^{N-1} \left(\{(k + 1)\alpha\} - \frac{1}{2} \right) \leq \frac{3}{2} \sum b_i$ where $\sum b_i q_i$ is the Ostrowski representation of $N \geq 0$.*

Corollary 2.5.6. *For any irrational α and real x there are infinite integers n satisfying $|S_n(x, \alpha)| \leq \frac{3}{2}$.*

Proof. This follows immediately from the previous corollary, since each integer q_i has Ostrowski representation $1.q_i$, so that $\sum b_i = 1$ and $\left|S_{q_i}(\alpha)\right| \leq \frac{3}{2}$ □

We now proceed to use the results of theorem 2.5.4 to derive results on the growth of peak values.

2.5.3 Peak growth derived from the pointwise results

We now relate our bounds to the number-theoretic bounds obtained by Lang.

First note that if $N = \sum_{i=0}^{p} b_i q_i$ then, by definition, $0 \leq b_i \leq q_{i+1}/q_i$ and so $O(n) = \sum_{i=0}^{p} b_i \leq \sum_{i=0}^{p} q_{i+1}/q_i$. Now recall that if α is of quasiperiodic type f, there is an $M \geq 0$ so that $q_{i+1}/q_i \leq f(q_i)$ for $i \geq M$. Put $B = \sum_{i=0}^{M} b_i$, then for $p > M$ and using $f \geq 1$ and increasing (so $f(N) \geq f(q_p)$ we have:

$$O(N) = \sum_{i=0}^{p} b_i \leq B + \sum_{i=M+1}^{p} f(q_i) < B + (p-M)f(q_p) \leq B + (p-M)f(N)$$

We need an estimate for p. Using equation 1.9 we have $q_n \geq 2^{n/2}$ for $n \geq 2$. Since the definition of p is $q_p \leq N < q_{p+1}$ we get $2^{p/2} \leq N$ or $p \leq \lfloor 2\log N/\log 2 \rfloor$. Hence our peak growth estimate becomes

$$|S_N(\alpha)| < \frac{3}{2}\left(B + \frac{2\log N}{\log 2} f(N)\right) = O\left(\log N\, f(N)\right) \tag{2.6}$$

Note that that we have not needed here the Lang requirement that $f(t)/t$ be decreasing. Also we have used a coarse bound for p which could potentially be improved, but which suits our purposes at this point.

We now compare our estimate with the Lang's estimate of $O\left(\int_1^N f(t)/t\, dt\right)$ for our two classes of section 2.3.1. First, if α is of constant type ($f = const$) we get:

$$S_N(\alpha) = O(\log N) \tag{2.7}$$

which coincides with Lang. If we take α to be of the class (of almost all numbers) of supra-log type $f(t) = (\log t)^{1+\epsilon}$ for $\epsilon > 0$ we get

$$S_N(\alpha) = O(\log N)^{2+\epsilon}$$

which again coincides with the Lang estimate.

Chapter 3

The Knill sum of cotangents

In this chapter we study the quasiperiodic sum defined by the value function $f(x) = \cot \pi x$, namely $S_n(x, \alpha, f) = \sum_{k=0}^{n-1} \cot \pi(x + k\alpha)$. This value function is neither bounded nor integrable (though it does have a principal value). The main result of this chapter is:

Theorem. *Let α be irrational with quasiperiods q_n and approximation errors $\alpha_n = \|q_n\alpha\|$, then for $n \geq 2$:*

$$\left|\sum_{r=1}^{q_n} \cot \pi r\alpha\right| < \frac{1}{\pi\alpha_n} + \frac{\pi\alpha_n}{2} \tag{3.1}$$

When α is of constant type c this becomes:

$$\left|\sum_{r=1}^{q_n} \cot \pi r\alpha\right| < \frac{cq_n}{\pi} + \frac{\pi}{2q_{n+1}}$$

3.1 Existing results

There has been surprisingly little study of this sum for irrational α, although there are a few related results. Hardy & Littlewood famously studied $S_n^*(\alpha) = \sum_{r=1}^{n} \csc \pi r\alpha$ in 1930 [23] using techniques of complex analysis, and establishing $S_n(\alpha) = O(n)$ for α of constant type.

In 2007 Sinai & Ulcigrai studied $S_n^{**}(x, \alpha) = \frac{1}{n} \sum_{r=1}^{n}(1 - \exp 2\pi i(x + r\alpha))^{-1}$ [52] and showed that this has a limiting distribution. Building on the techniques in 2009 [53] they were able to prove the Hardy & Littlewood result using elementary methods only. In addition (again using elementary methods) they showed for α of constant type that $\frac{1}{q_n} \sum_{r=1}^{q_n} \sum_{s=1}^{q_n} \exp 2\pi i(rs\alpha) = O(1)$. Since we will use only elementary methods also, it would be interesting to compare these various methods in future work.

Knill seems to have been the first to study the sum $S_n(\alpha) = \sum_{r=1}^{n} \cot \pi r\alpha$ directly. In 2012 he studied the particular base case $S_n(\omega) = \sum_{k=1}^{n} \cot \pi k\omega$ at the golden rotation $\omega = \left\{\frac{1}{2}\left(\sqrt{5} + 1\right)\right\}$. It was the object of study of a paper[31] (published to date only on the arxiv), and the third in a series of papers, the first with Lesieutre [32] and then Tangerman [33]. These were originally motivated by a problem in critical KAM theory resulting in the quasiperiodic sum $S_n^*(x,\alpha) = \sum_{k=0}^{n-1} \log(2 - 2\cos 2\pi(x+k\alpha))$ for general α. (Note that this sum is the logarithm of the Sudler product introduced in section 1.1.2.) This leads naturally to the Knill sum via the derivative $\frac{d}{dx} S_n^*(x,\alpha) = 2\pi S_n(x,\alpha)$.

The main result of the third paper is to show that the graphs of section 1.1.2.1 converge along odd and even values of m. This is achieved by the ingenious move of studying the sequence of value functions $s_{2n}(x) = \frac{1}{q_{2n}} S_{\lfloor q_{2n} x \rfloor}(\omega)$, and finding that the sequence converges pointwise to a bounded value function $s(x)$ continuous on the right and satisfying $s(\omega x) = -\omega s(x)$. (The sequence s_{2n+1} converges to $-s$).

This gives us immediately that, for $k < q_n$, $|S_k(\omega)| \le q_n \|s_n\|$ (using the supremum norm $\|s_n\| = \sup_{x\in\mathbb{T}} |s_n(x)|$), and hence for $q_{n-1} \le k < q_n$ we have $|S_k(\omega)|/k \le (q_n/q_{n-1}) \|s_n\|$. But ω is of constant type meaning that (q_n/q_{n-1}) is bounded, and also $|s_n| \to |s|$, so that $(q_n/q_{n-1}) \|s_n\|$ is bounded. It follows that $S_k(\omega) = O(k)$.

3.2 New results

Our work is somewhat complementary to that of Knill: we use a different approach and this gives us bounds on $|S_k(\alpha)|$ for almost all α rather than just $\alpha = \omega$, but only for the cases $k = q_n$. Possibly either method could be extended to give results for almost all α *and* all positive integers k.

Our objective is to establish bounds for $S_{q_n}(\alpha) = \sum_{k=1}^{q_n} \cot \pi k\alpha$. Let p_n/q_n be the nth convergent of α, so that $q_n\alpha - p_n = (-1)^n \alpha_n$ with $0 < \alpha_n < 1/q_{n+1}$.

For $n \ge 2$, let $r_p = rp_n \bmod q_n$ so that, as r runs through the residues $1 \dots q_n - 1$, so does r_p.

We consider n even so that $\alpha = (p_n + \alpha_n)/q_n$. It follows that the fractional part $\{r\alpha\}$ lies in the interval $(r_p/q_n, (r_p+1)/q_n)$. Note that when $x, x+\theta \in (0,\pi)$ and $\theta > 0$ we have $\cot x > \cot(x+\theta)$, and hence $\cot \pi r_p/q_n > \cot \pi r\alpha$. We now use the fact that $\sum_{r=1}^{q-1} \cot \pi r/q = 0$ to obtain:

$$\sum_{r=1}^{q_n-1} \cot \pi r\alpha < \sum_{r=1}^{q_n-1} \cot \pi \frac{r_p}{q_n} = 0 \tag{3.2}$$

Similarly we have $\cot \pi r\alpha > \cot \pi(r_p+1)/q_n)$, but now $r_p + 1$ runs through the values $2 \dots q_n$. The particular value $r_p = q_n - 1$ results in a singularity of $\cot \pi(r_p+1)/q_n$ so we must treat this case separately. Let r^* be the solution of the residue equation $rp_n = q_n - 1 \bmod q_n$. We will establish a small lemma to help calculate r^*.

Lemma 3.2.1. *For $n \ge 1$ the inverse of the residue p_n* mod q_n *exists and lies in the residue class of $(-1)^{n-1} q_{n-1}$.*

Proof. From equation 1.7 we use $p_{n+1}q_n - p_nq_{n+1} = (-1)^n$ together with equation (1.8) $q_{n+1} = a_nq_n + q_{n-1}$ to give us $-p_nq_{n-1} \equiv (-1)^n \bmod q_n$ and so $p_n(-1)^{n-1}q_{n-1} \equiv 1 \bmod q_n$. □

Hence, for n even, the residue inverse to p_n is $q_n - q_{n-1}$, and so $r^* \equiv (q_n - q_{n-1})(q_n - 1) \equiv q_{n-1} \bmod q_n$. This gives us, again using $\sum_{r=1}^{q-1} \cot \pi r/q = 0$, and then using from (1.6) $q_{n-1}\alpha - p_{n-1} = (-1)^{n-1}\alpha_{n-1}$:

$$\begin{aligned}
\sum_{r=1}^{q_n-1} \cot \pi r\alpha &> \sum_{r_p=1}^{q_n-2} \cot \pi \frac{r_p+1}{q_n} + \cot \pi r^*\alpha \\
&= \left(-\cot \frac{\pi}{q_n}\right) + \cot \pi q_{n-1}\alpha \\
&= -\cot \frac{\pi}{q_n} - \cot \pi\alpha_{n-1} \text{ for even } n \geq 2 && (3.3)\\
&> -\frac{1}{\pi}\left(q_n + \frac{1}{\alpha_{n-1}}\right) \text{ for even } n \geq 2 && (3.4)
\end{aligned}$$

But also for n even, $\{q_n\alpha\} = \alpha_n > 0$ so that we can use the expansion $\left(1 - x^2/2\right)/x < \cot x < 1/x$ for $0 < x < 1$ to obtain:

$$\frac{1}{\pi\alpha_n}\left(1 - \frac{(\pi\alpha_n)^2}{2}\right) < \cot \pi q_n\alpha < \frac{1}{\pi\alpha_n} \tag{3.5}$$

Adding (3.5) to inequalities (3.2) and (3.4) to obtain for even $n \geq 2$, that:

$$\frac{1}{\pi\alpha_n}\left(1 - \frac{(\pi\alpha_n)^2}{2}\right) - \frac{q_n}{\pi} - \frac{1}{\pi\alpha_{n-1}} < \sum_{r=1}^{q_n} \cot \pi r\alpha < \frac{1}{\pi\alpha_n} \tag{3.6}$$

Denoting the sum on the LHS by $\mathcal{F}_n$ gives us:

$$\mathcal{F}_n = \frac{1}{\pi\alpha_n}\left(1 - \frac{(\pi\alpha_n)^2}{2} - q_n\alpha_n - \frac{\alpha_n}{\alpha_{n-1}}\right)$$

From equation (1.10) we get $q_n\alpha_n < q_n/q_{n+1} < 1$. Also $\alpha_n/\alpha_{n-1} < 1$ and so $\mathcal{F}_n > \left(-1 - (\pi\alpha_n)^2/2\right)/\pi\alpha_n$. This gives us in (3.6):

$$\left|\sum_{r=1}^{q_n} \cot \pi r\alpha\right| < \frac{1}{\pi\alpha_n} + \frac{\pi\alpha_n}{2}$$

A completely analogous argument holds for n odd, giving us finally, for any $n \geq 2$:

Theorem 3.2.2. *Let α be irrational with quasiperiods q_n and approximation errors $\alpha_n = \|q_n\alpha\|$, then for $n \geq 2$:*

$$\left|\sum_{r=1}^{q_n} \cot \pi r\alpha\right| < \frac{1}{\pi\alpha_n} + \frac{\pi\alpha_n}{2} \tag{3.7}$$

We can now apply this result to the classes of α introduced in section 1.3.3, using from (1.6) $\alpha_n < 1/q_{n+1}$:

Corollary 3.2.3. *If a real number α is of constant type so that $\alpha_n > 1/cq_n$ for some constant c, then:*

$$\left|\sum_{r=1}^{q_n} \cot \pi r\alpha\right| < \frac{cq_n}{\pi} + \frac{\pi}{2q_{n+1}} = O(q_n)$$

Also, since for almost all real numbers α, α is of supra-log type with $\alpha_n \geq 1/q_n(\log q_n)^{1+\epsilon}$ for $n > N(\alpha)$, we also have for almost all real α and $n > N(\alpha)$:

$$\left|\sum_{r=1}^{q_n} \cot \pi r\alpha\right| < \frac{1}{\pi} q_n(\log q_n)^{1+\epsilon} + \frac{\pi}{2q_{n+1}} = O\left(q_n(\log q_n)^{1+\epsilon}\right)$$

In the special case of the golden rotation $\alpha = \omega$ (which is of constant type), we have $q_n = F_{n+1}$, and $\alpha_n = \omega^{n+1}$, giving us:

Corollary 3.2.4. *For the golden rotation $\omega = \frac{1}{2}\left(\sqrt{5}-1\right)$ and $n \geq 3$ we have:*

$$\left|\sum_{r=1}^{F_n} \cot \pi r\omega\right| < \frac{1}{\pi\omega^n} + \frac{\pi\omega^n}{2} \tag{3.8}$$

Chapter 4

The Sudler product of sines

In this chapter we study the quasiperiodic product defined by the value function $f(x) = 2\sin\pi x$ (see section 1.1.2). This gives us, for the rotation number α:

$$P_n(x) = \prod_{k=0}^{n-1} 2\sin\pi\{x + k\alpha\} = \left|\prod_{k=0}^{n-1} 2\sin\pi(x + k\alpha)\right|$$

We are also interested in its additive form $S_n(x, \alpha, \log f) = \log P_n(x, \alpha, f) = \sum_{k=0}^{n-1} \log|2\sin\pi(x + k\alpha)|$. Our previous chapters have studied only sums, and so it may seem surprising that we are switching to study the product of this value function rather than its sum $S_n(x, \alpha, f) = \sum_{k=0}^{n-1} |2\sin\pi(x + k\alpha)|$ (note that $S_n(x, \alpha, \log f)$ and $S_n(x, \alpha, f)$ are very different functions). The reason is pragmatic: the product (and its additive form) occur in a remarkable variety of mathematical contexts whereas the sum does not; the product has therefore taken precedence in our research.

Our main goal in this chapter is to establish a foundational result for the case of the golden ration $\alpha = \omega$ (where q_n is a quasiperiod for ω and is in fact a Fibonacci number):

$$Q_n = P_{q_n}(\omega) = \left|\prod_{r=1}^{F_n} 2\sin\pi r\omega\right| \to c > 0 \tag{4.1}$$

ie for the golden rotation $\alpha = \omega$, the values of the product over a quasiperiod converge to a strictly positive value (computation indicates it is approximately 2.407). This innocuously simple result will require some work to establish, but is the foundation for developing our other new results.

The Sudler product seems intrinsically harder to analyse than the previous sums (chapters 2,3) even though they are closely related. In fact the analysis of this product turns out to require the results we previously obtained for both the sums. As with the value function of the Knill sum (chapter 3), the additive form of the value function (namely $\log f(x) = \log|2\sin\pi x|$) is not bounded, and so the Denjoy-Koksma inequality cannot be used directly. However the function is integrable, and Knill & Lesieutre[32] developed a modified approach in which a surgery technique is applied to the singularity. This seems a promising approach but unfortunately the results obtained are suboptimal (see below). As Lubinsky was led to comment[41], this is a problem area for which classical approaches "yield essentially weaker results".

Our study is here limited to the special case of the the golden rotation $\alpha = \omega$, and the initial condition of $x = \alpha = \omega$. Even with these significant restrictions, this chapter is much longer than the previous ones, as the proof breaks down into a number of parts each requiring some detailed analysis.

4.1 Existing results

Practically all work to date on this function has studied the simplified case of $x = \alpha$, ie the case $P_n(\alpha) = \prod_{k=1}^{n} |2 \sin \pi(k\alpha)|$. In this form it arises in a surprising number of fields of pure and applied mathematics and physics, but disguised by a number of different representations and terminologies. We have already mentioned a number of these in our survey of application areas of quasiperiodic sums and products (section 1.2), but in this section we give a consolidated summary of results connected specifically with the Sudler product.

In pure mathematics there are important applications in partition theory ([55, 59]), Padé approximation and q-series (see [41] for a list of 13 examples), whereas in applied mathematics the function has been studied in connection with strange non-chaotic attractors (SNAs) and critical KAM theory (see [20, 37, 33, 1, 31] for examples). In the context of SNAs (our own area of interest) the products arise in the renormalisation analysis of skew products.

The common problem in each of these areas is to understand some aspect of the growth of the sequence $P_n(\alpha)$ with n.

This list of application areas is no doubt incomplete if only because of the remarkable range of representations and terminology under which the function appears. The representation $P_n(\alpha) = \prod_{r=1}^{n} |2 \sin \pi r\alpha|$ arises in dynamical systems as the absolute value $|z|$ in the skew product $(\theta, z) \longmapsto (\theta + \alpha, 2z \sin \pi\theta)$ (with initial condition $(\alpha, 1)$). However putting $z = \exp(2i\pi\alpha)$ we obtain the representation $P_n(\alpha) = \prod_{r=1}^{n} |1 - z^r|$ which is the modulus of the restricted Euler function, and links us to partition theory (amongst other things). Further if we take the q−Pochhammer symbol $(a; q)_n$ and put $a = q = z = \exp(2i\pi\alpha)$ we have $P_n(\alpha) = |(z; z)_n|$ which links us to q−series and string theory. Finally we have $\log P_n(\alpha) = S_n(\alpha)$, which is a quasiperiodic sum with value function $\log f(x) = \frac{1}{2}\log(2 - 2\cos 2\pi x)$. The latter function is a constant multiple of the Hilbert transform of the remainder value function $\pi f^{\text{Rem}}(x) = \pi \left(\{x\} - \frac{1}{2}\right)$ (see [32]), and its derivative is a constant multiple of the Knill value function $f^{\text{Kni}}(x) = \cot \pi x$. The sum $S_n(\alpha)$ is also called the

Birkhoff sum of f and has been studied in ergodic theory and KAM theory.

4.1.1 Growth with n of the norm $\|P_n(\alpha)\| = \sup_\alpha |P_n(\alpha)|$

The first in-depth study of the function $P_n(\alpha)$ seems to have been made by Sudler in 1964[55], although Erdős and Szekeres previously stated a "very easy" result (without proof) in 1959[11][1]. Sudler[2] showed that in the limit the norm grows exponentially[3] with n, with the growth rate E being given by the formula:

$$E = \lim_{n\to\infty} \|P_n(\alpha)\|^{1/n} = \alpha_0^{-1} \int_0^{\alpha_0} \log|2\sin\pi\alpha|\, d\alpha = 1.2197\ldots$$

where α_0 is the (unique) solution in $[1/2, 1]$ of $\int_0^{\alpha_0} \alpha \cot \pi\alpha \, d\alpha = 0$. Further he also showed that $\|P_n(\alpha)\|$ is achieved at α_n where $\alpha_n \sim \alpha_0/n$ as n grows.

Freiman and Halberstam (1988) [17] later provided an alternate proof which gives the same result in the even more elegant form $E = 2\sin\pi\alpha_0$ (where α_0 is as above). (Incidentally this seems to be the first paper to study the function $P_n(\alpha)$ as a first-class citizen, ie worthy of study in its own right rather than as as a stepping stone to the estimation of other functions).

In 1998 Bell et al [5] proved a number of stronger results, in particular that the norm of the sub-product $\left\|\prod_1^n 2\sin r^k\alpha\right\|$ grows exponentially for any $k \geq 1$. More recently Jordan Bell (2013) [7] adapted the method of Wright [59] to show $\|P_n(\alpha)\| \sim C_1\sqrt{n}E^n$, and also generalised the result to the L_p-norm: $\|P_n(\alpha)\|_p = \left(\int_0^1 P_n(\alpha)^p d\alpha\right)^{1/p} \sim C_1 \left(C_2 n^{-3/2}\right)^{1/p} \sqrt{n}E^n$ for calculated constants C_1, C_2.[4]

4.1.2 Growth of peaks of the sequence $P_n(\alpha)$ at fixed α

We might guess that since the norm of the function $P_n(\alpha)$ grows exponentially, then the pointwise growth rate (the growth rate of the sequence $P_n := P_n(\alpha)$ at a fixed value of α) would also be exponential. However this turns out not to be the case. Using the theory of uniform distribution, Lubinsky [42] showed that for almost all α, $\lim_{n\to\infty} (P_n(\alpha))^{1/n} = 1$, ie the growth is ae sub-exponential, not exponential. This apparent conflict is explained by Figure 4.1.1 in which we see that the exponential growth of the norm is achieved at a peak which is uncharacteristic of the rest of the function. This peak narrows and converges on 0 as n grows, so that for any fixed value of α the peak will pass it for some value of n, after which the growth at that point will revert to being sub-exponential.

[1] Their claim was that $\lim_{n\to\infty} \|P_n(\alpha)\|^{1/n}$ exists and lies between 1 and 2. Sudler found the limit precisely.

[2] Freiman and Halberstam (1988) attribute this result to Wright[17] but from a careful reading of both papers [55, 59] it seems that Sudler has priority. Sudler does however acknowledge the help of Wright as a referee in improving the proofs, and Wright also improves Sudler's result in his own subsequent paper.

[3] More precisely he showed $\|P_n(\alpha)\|^{1/n} = E + O(\log n/n)$ where E is the constant above.

[4] C_2 is actually $O(p^{-1/2})$, but is independent of n.

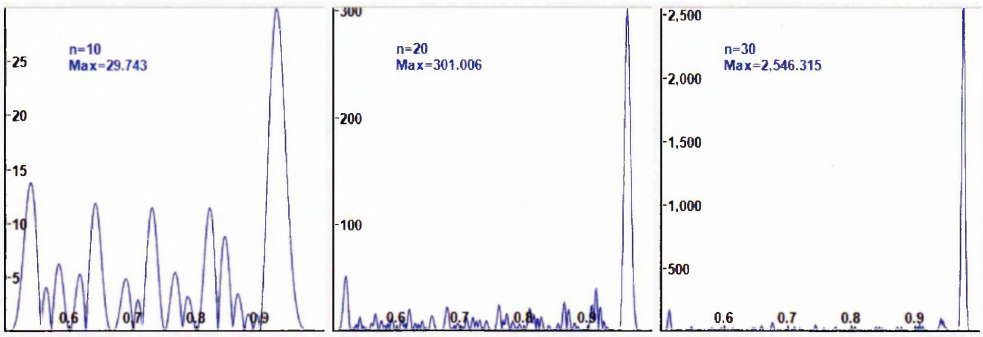

Figure 4.1.1: $P_n(\alpha)$ plotted over the interval $\alpha \in [0, 0.5]$ for various values of n. Note how the norm is achieved at a point which is converging on the origin, but that the exponential growth achieved there is uncharacteristic.

In [32] Knill and Lesieutre adapted Herman's Denjoy-Koksma inequality [25] to show that, for some constant $C(\alpha)$, $P_n(\alpha) < n^{Cn^{1-1/(1+s)}\log n}$ when α is of type $f(t) = ct^s$ with $s \geq 0, c \geq 1$.

Lubinsky [41] studied the problem in the context of q-series and showed that, for almost all α, and all $\epsilon > 0$, there are constants $N, C_1, C_2 \geq 0$ (dependent only on α, ϵ) such that for all $n \geq N$ we have $n^{-C_1(\log n)^\epsilon} \leq P_n(\alpha) \leq n^{C_2(\log n)^\epsilon}$, and further that the inequality improves to $n^{-C_1} \leq P_n(\alpha) \leq n^{C_2}$ for all α of constant type. Amongst other results he also showed that, for all irrational α, $\limsup_n P_n(\alpha)/n \geq 1$ (from which we can deduce $C_2 \geq 1$ above), and was strongly of the opinion that for all α, $\liminf P_n(\alpha) = 0$. He established that the latter result certainly holds for α of increasing type.

4.1.3 Growth of peaks at the golden rotation

The results of the previous section bound the norm growth of $P_n(\omega)$ with n. However, as with the previous cases we have studied, the peaks of this function are very different from its values elsewhere (see section 1.1.2). In certain applications, and in particular in the study of strange non-chaotic attractors (SNAs), we require a sharper pointwise estimate of the size of $P_n(\omega)$ at every point n.

Working in the context of KAM theory, Knill and Tangerman studied the quasiperiodic sum $S_n^*(\omega) = \sum_{r=1}^n \log(2-2\cos 2\pi r\omega)$ in their 2011 paper [33]. (In fact $S_n^*(\alpha) = 2S_n(\alpha) = 2\log P_n(\alpha)$). Note that the convergents p_n/q_n of the golden rotation case are in fact F_n/F_{n+1} where F_n is the nth Fibonacci number (indexed from $F_0 = 0$). Taking the sequence (F_n) as a renormalisation (decimation) scale, they presented experimental graphical and numerical evidence for the existence of an asymptotic renormalisation function. The renormalisation approach was also earlier studied by Kuznetsov et al (1995) [37] in a slightly more general setting, where they used polynomial approximation to obtain strong numerical evidence also for asymptotic renormalisation functions.

Assuming the existence of this asymptotic function as a hypothesis, Knill and Tangerman deduced the following consequences:

Theorem 4.1.1 (Knill & Tangerman). *Consequences of the (not proven) hypothesis (where ω is the golden rotation):*

1. $S_n(\omega)$ tends to a constant along the decimation subsequence $n = F_m$, ie $S_{F_m}(\omega) \longrightarrow c$ for some constant c.

2. The sequence $S_n(\omega)/\log n$ has accumulation points at (a) 0 and (b) 2.

3. The sequence $S_n(\omega)/\log n$ is bounded[5].

In 2012 Knill [31] studied a related sum of cotangents $S_n'(\omega) = \sum_{r=1}^{n} \cot \pi r\omega$, and demonstrated that this sum has marked self-similarity. From this he also sketched how one might derive the results above. We will give a detailed proof of the slightly stronger results which are set out below in terms of $P_n(\omega)$. Note that $S_n(\omega) = \sum_{r=1}^{n} \log(4\sin^2 \pi r\omega) = 2\log P_n(\omega)$ and from this it is easily seen that theorem 4.1.2 below implies theorem 4.1.1 above. (Note that (2) above is the logarithmic equivalent of both (1),(2) below). In addition the approach is complementary to those of Knill and Lubinsky, and in the next chapter (5) we will show that it can be extended to obtain improved results.

Theorem 4.1.2. *The following results hold for the golden rotation number ω:*

1. There is a constant $c > 0$ such that $P_{F_n}(\omega) \longrightarrow c$. This is equivalent to 1 & 2a above.

2. For the same constant c, $P_{F_n-1}(\omega)/F_n \longrightarrow c/(2\pi\sqrt{5})$. This is a slightly stronger result than 2b.

3. There are real constants $C_1 \le 0 < 1 \le C_2$ and $N \ge 0$ such that $n^{C_1} \le P_n(\omega) \le n^{C_2}$ for $n \ge N$. This is equivalent to 3 above.

The proof of the first and foundational result, namely that $P_{F_n}(\omega) \longrightarrow c$ for some constant c, will occupy the bulk of this chapter. In subsection 4.7 we will deduce results (2) and (3).

4.2 Overview of the proof

At the end of the previous section we described how Knill and Tangerman recently presented experimental graphical and numerical evidence for the existence of an asymptotic renormalisation function when $\omega = \left(\sqrt{5}-1\right)/2$. From this they deduced three consequences. However we will show in section 4.7 that the second and third consequences flow directly from the first, and have no

[5] Lubinsky [41] proved this result for all irrational α of constant type

dependency on the experimental function. Our main goal now is to establish the first consequence, namely that the sequence $P_{F_n}(\omega)$ converges to a constant.

This rather simple statement belies the surprising amount of work that seems necessary to prove it. However it is worth noting that both Knill and Lubinsky remark that this is one problem area where established procedures and powerful tools fall short. This has also been our own experience, and we have felt very much forced back to a proof from first principles.

Using renormalisation terminology, we "decimate" the sequence $P_n(\omega) = \left|\prod_{r=1}^{n} 2\sin \pi r\omega\right|$ by picking every F_nth element to yield a "renormalisation sub-sequence" $Q_n = \left|\prod_{r=1}^{F_n} 2\sin \pi r\omega\right|$. Our main result is now the following:

Theorem 4.2.1. *The sequence* $Q_n = \left|\prod_{r=1}^{F_n} 2\sin \pi r\omega\right|$ *converges to a strictly positive limit* $c > 0$

(Computation indicates $c \approx 2.407\ldots$).

The proof of this theorem will occupy the main body of this chapter (Sections 4.2 – 4.6).

In section 4.7 we deduce from the main result the two other results reported by Knill and Tangerman. In particular this includes the result that the Sudler product growth at ω is bounded by a power law. Knill and Tangerman suggested that this particular result would flow from a modification of the proof of the Denjoy-Koksma result in ergodic theory, but on closer examination further work appears necessary. We have again found the need to derive this corollary from first principles.

In section 4.4 we introduce a core strategy which is to exploit the continued fraction convergents to the inverse golden mean ω. These convergents are the ratios of subsequent Fibonacci numbers F_{n-1}/F_n, and $\omega = (F_{n-1} - (-\omega)^n)/F_n$ (see (4.7)). This gives us:

$$Q_n = \left|\prod_{r=1}^{F_n} 2\sin \pi r\omega\right| = \left|\prod_{r=1}^{F_n} 2\sin\left(\frac{\pi r\,(F_{n-1} - (-\omega)^n)}{F_n}\right)\right|$$

This allows us to develop a representation of Q_n as a product of three rather more tractable products, namely:

$$Q_n = A_n\, B_n\, C_n = (2F_n \sin \pi\omega^n)\left(\prod_{t=1}^{F_n-1} \frac{s_{nt}}{2\sin \pi\frac{t}{F_n}}\right)\prod_{t=1}^{F_n-1}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right)^{1/2}$$

where $s_{nt} = 2\sin \pi\,(t/F_n - \omega^n\,(t_n - 1/2))$ and t_n is the fractional part of $(F_{n-1}t)/F_n$.

It is easy to show that $A_n \to 2\pi/\sqrt{5}$. The products B_n, C_n also converge to strictly positive limits, but the latter demonstrations require significantly greater effort, and receive their own sections.

In section 4.5 we shall deal with the convergence of the simpler of the two products, namely $C_n = \prod_{t=1}^{F_n-1}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right)^{1/2}$. In section 4.6 we shall deal with the convergence of $B_n = \left(\prod_{t=1}^{F_n-1} \frac{s_{nt}}{2\sin \pi t/F_n}\right)$.

This requires the most work and is broken down into several significant sub-sections.

4.3 Preliminaries

4.3.1 Conventions for sums and products

As usual we will define the empty sum to have the value 0, and the empty product to have the value 1.

Given a summable sequence (a_r) we will find it useful to define a generalised summation notation $\sum_{r=x}^{y} a_r$ to include real (rather than integer) upper and lower bounds x, y. We do this by defining the step function $f(t) = a_r$ for $t \in [r, r+1)$, and then

$$\sum_{r=x}^{y} a_r = \int_x^y f(t)\mathrm{d}t \tag{4.2}$$

If $\log f$ in integrable on $[x, y]$ we define the multiplicative analogue as

$$\prod_{r=x}^{y} a_r = \exp \int_x^y \log f(t)\mathrm{d}t \tag{4.3}$$

For example, for odd integers $n = 2k + 1$:

$$\begin{aligned} \sum_{1}^{n/2} a_r &= \sum_{1}^{k} a_r + \frac{1}{2} a_k \\ \prod_{1}^{n/2} a_r &= \prod_{1}^{k} a_r \times a_k^{1/2} \end{aligned}$$

Note that for integer x, y the definitions coincide with normal summation and product notation.

4.3.2 Results on Fibonacci numbers

We will make use of some standard results about Fibonacci numbers $F_n = (0, 1, 1, 2, 3, 5, 8 \ldots)$ defined for $n \geq 0$ by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \tag{4.4}$$

$$F_n \text{ is even iff } n = 3k \text{ for some } k \geq 0 \tag{4.5}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\omega^{-n} - (-\omega)^n\right) = \frac{\omega^{-n}}{\sqrt{5}} + O(F_n^{-1}) \tag{4.6}$$

$$F_n\omega = F_{n-1} - (-\omega)^n \Leftrightarrow \omega = \frac{F_{n-1}}{F_n} - \frac{(-\omega)^n}{F_n} = \frac{F_{n-1}}{F_n} + O(\omega^{2n}) \tag{4.7}$$

4.3.3 Inequalities

We gather here various inequalities which we will need during the main proofs.

Lemma 4.3.1. *For x in $(0, \pi/2)$ we have $2x/\pi < \sin x < x$.*

Proof. The derivative of $f(x) = x - \sin x$ is $1 - \cos x$ which is positive. So $f(x)$ is increasing, and $f(0) = 0$, and the right side inequality follows. For the left side we use the fact that $\sin x$ is convex in this interval and hence lies above the line segment joining $(0, 0)$ and $(\pi/2, 1)$. But this is $2x/\pi$. □

Lemma 4.3.2. *For $n \geq 2$, let (a_t) $t = 1 \ldots n$ be a sequence of real numbers satisfying $|a_t| < 1$ with $A = \sum |a_t| < 1$.*

$$1 - A < \prod_{t=1}^{n}(1 + a_n) < e^A < \frac{1}{1 - A}$$

Proof. $\prod_{t=1}^{n}(1 + a_t) \geq \prod_{t=1}^{n}(1 - |a_t|)$. Then $\prod_{t=1}^{n}(1 - |a_t|) > 1 - A$ is clearly true for $n = 2$ and the left hand side of the result follows by induction.

Also $\prod_{t=1}^{n}(1 + a_t) \leq \prod_{t=1}^{n}(1 + |a_t|) < \prod_{t=1}^{n} e^{|a_t|} = e^A$.

The final part follows from comparing the Taylor series for the two expressions. □

4.3.4 Special sequences used in this chapter

In addition to the Fibonacci sequence $(F_i) = (0, 1, 1, 2, 3, 5, 8 \ldots)$, we make extensive use of a number of derived sequences which we define here for convenience. Note we only define them for integers $n \geq 1, t \geq 0$.

$$s_{nt} = 2\sin\pi\left(\frac{t}{F_n} - \omega^n\left(\left\{\frac{tF_{n-1}}{F_n}\right\} - 1/2\right)\right) \tag{4.8}$$

$$\xi_{nt} = \begin{cases}\left\{\frac{tF_{n-1}}{F_n}\right\} - \frac{1}{2} & (t \not\equiv 0 \bmod F_n)\\ 0 & (t \equiv 0 \bmod F_n)\end{cases} \tag{4.9}$$

$$\xi_{\infty t} = \{t\omega\} - \frac{1}{2} \tag{4.10}$$

$$h_{nt} = \cot\frac{\pi t}{F_n}\sin(\pi\omega^n\xi_{nt}) \ (t \not\equiv 0 \bmod F_n) \tag{4.11}$$

Note that $s_{nt} = 2\sin\pi\,(t/F_n - \omega^n\xi_{nt})$ when $t \not\equiv 0 \bmod F_n$, but not when $t \equiv 0 \bmod F_n$ due to the alternative definition of ξ_{nt}. This reflects the fact that the two sequences play very different roles, and each definition makes sense in its own context. We have also chosen to leave h_{nt} undefined for $t \not\equiv 0 \bmod F_n$.

Lemma 4.3.3. *For the sequences $s_{nt}, \xi_{nt}, \xi_{\infty t}$ defined above:*

1. *For fixed $n \geq 1$, the sequences $|s_{nt}|, \xi_{nt}, h_{nt}$ are periodic sequences of period F_n, and further s_{nt}, ξ_{nt} are both odd sequences in t (ie of the form $a_t = -a_{-t}$) and h_{nt} is an even sequence in t (ie of the form $a_t = a_{-t}$).*

2. *Both $|\xi_{nt}| < 1/2$ and $|\xi_{\infty t}| < 1/2$ with the exception of $\xi_{\infty 0} = -1/2$.*

3. *In the range $0 \leq t \leq F_n - 1$, $s_{nt} \geq s_{n0} > 0$ with equality only at $t = 0$. For any t, $s_{n,F_n+t} = -s_{nt}$*

4. *$\xi_{n,F_n-t} = -\xi_{nt}$ whereas $s_{n,F_n-t} = s_{nt}$, and $h_{n,F_n-t} = h_{nt}$*

5. *For $1 \leq t \leq F_{n-1}$ we have $\xi_{nt} - \xi_{\infty t} = t(-\omega)^n/F_n = O(\omega^n)$ and $\lim_{n\to\infty} \xi_{nt} = \xi_{\infty t}$*

Proof. □

1. Note that $\{tF_{n-1}/F_n\}$ is of period F_n, and the periodicity results follow, noting also that $|\sin \pi x|$ is of period 1. Also we have $\{-x\} = 1 - \{x\}$, from which the oddness of ξ_{nt} immediately follows. The oddness of s_{nt} then follows from the oddness of $\sin x$. The evenness of h_{nt} follows from the oddness of both cot and sin.

2. Both results follow from $0 < \{x\} < 1$ unless $x = 0$. But $\{tF_{n-1}/F_n\} = 0$ only for $t \not\equiv 0 \bmod F_n$ and then $\xi_{nt} = 0$. And $\{t\omega\} = 0$ only for $t = 0$.

3. For $t = 0$ we have $s_{nt} = s_{n0} = 2\sin \pi\omega^n/2 > 0$. For $n = 1, 2$ the only possibility is $t = 0$, but for $n \geq 3$ and $1 \leq t \leq F_n - 1$, then $s_{nt} \geq s_{n1} = 2\sin \pi \left(F_n^{-1} - \omega^n \xi_{1t}\right)$. But $|\xi_{1t}| < 1/2$, and $F_n^{-1} = \sqrt{5}\omega^n/(1 - (-1)^n\omega^{2n}) > 2\omega^n$ so that $s_{n1} > s_{n0} > 0$. The second part follows by noting that substituting $F_n + t$ in s_{nt} simply adds π to the argument of the sine function.

4. These now follow easily from the previous results.

5. Since $t \neq 0$, we have $\xi_{nt} - \xi_{\infty t} = \{tF_{n-1}/F_n\} - \{t\omega\}$. Now by (4.7) $t\omega = tF_{n-1}/F_n - t(-\omega)^n/F_n$, but $t < F_n$ so $|t\omega - tF_{n-1}/F_n| < \omega^n < 1/F_n$ which means $\{t\omega\}$ is always inside the interval $\{tF_{n-1}/F_n\} \pm 1/F_n$, and we can deduce that $\xi_{nt} - \xi_{\infty t} = tF_{n-1}/F_n - t\omega = t(-\omega)^n/F_n$. The other results follow immediately.

4.4 The Decomposition $Q_n = A_nB_nC_n$

As described in section 4.2, we develop a decomposition of Q_n into a product of three other products, each of which converges to a positive constant. We shall prove the convergence of the first of these products within this section (as it is very straightforward), and the other two we shall deal with in subsequent sections.

Our central motivation here is to substitute the Fibonacci identity $\omega = (F_{n-1}/F_n) - (-\omega)^n/F_n$ (see (4.7)) into the definition of Q_n and hence express $|\prod 2\sin \pi r\omega|$ as a perturbation of the rational sine product $|\prod 2\sin \pi r(F_{n-1}/F_n)|$, the latter product being equal to F_n (see (4.23)). This reduces the problem to one of demonstrating that the perturbation function itself has suitable behaviour, and this proves equivalent to showing that the product $B_n C_n$ converges as n grows. However rather than treating $B_n C_n$ as a single product, it is simpler to prove separately that each of B_n and C_n converge.

The substitution above gives us $Q_n = |\prod 2\sin \pi r\,((F_{n-1}/F_n) - (-\omega)^n/F_n)|$ which is a perturbation of the argument in each term of $|\prod 2\sin \pi r(F_{n-1}/F_n)|$ by a delta of $-r(-\omega)^n/F_n$. The sum of these deltas is non-zero, but some of the techniques we shall use to prove the convergence of B_n, C_n require that the sum of the deltas is 0. Fortunately, as we shall see, we can fix this by re-basing the arguments to result in a delta of $\omega^n(r/F_n - 1/2)$ - which then provides a zero sum for the deltas. This is most economically achieved once and for all at the beginning of our proof, and will simplify later proofs at the cost introducing a non-intuitive first step below. However once done, we proceed to make the substitution for ω, and the decomposition then follows naturally.

Lemma 4.4.1. *For $n \geq 1$ and $s_{nt} = 2\sin\pi\left(t/F_n - \omega^n\left(\left\{\frac{F_{n-1}t}{F_n}\right\} - \frac{1}{2}\right)\right)$ we have $Q_n = \left|\prod_{r=1}^{F_n}\left(2\sin \pi r\omega\right)\right| = A_n\, B_n\, C_n$ where:*

$$A_n = 2F_n \sin \pi\omega^n \longrightarrow \frac{2\pi}{\sqrt{5}} \tag{4.12}$$

$$B_n = \left(\prod_{t=1}^{F_n-1} \frac{s_{nt}}{2\sin \pi \frac{t}{F_n}}\right) \tag{4.13}$$

$$C_n = \prod_{t=1}^{(F_n-1)/2}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) \tag{4.14}$$

We first deal with the convergence of A_n by observing that since $\omega < 1$, we have $A_n = 2F_n \sin \pi\omega^n \sim 2F_n\pi\omega^n$ and the result follows by (4.6).

We start the main proof by carrying out the step discussed above to re-base our perturbation deltas. First we exploit the symmetry of the sine function around $\pi/2$, observing that a change of variables $r \mapsto F_n - r$ gives us $\prod_{r=1}^{F_n-1}\left(2\sin \pi r\omega\right) = \prod_{r=1}^{F_n-1}\left(2\sin \pi(F_n - r)\omega\right)$ and hence for any $n \geq 1$, using the product of sines formula:

$$\begin{aligned} Q_n^2 &= (2\sin \pi F_n\omega)^2 \prod_{r=1}^{F_n-1}\left(2\sin \pi r\omega\right)\left(2\sin \pi(F_n - r)\omega\right) \\ &= (2\sin \pi F_n\omega)^2 \prod_{r=1}^{F_n-1} 2\left(\cos \pi(F_n - 2r)\omega - \cos \pi F_n\omega\right) \end{aligned} \tag{4.15}$$

We can now use identity (4.7) and the cosine double angle formula to obtain:

$$
\begin{aligned}
Q_n^2 &= (2\sin\pi\omega^n)^2 \prod_{r=1}^{F_n-1} 2(-1)^{F_{n-1}} \left(\cos\pi((-\omega)^n + 2r\omega) - \cos\pi(-\omega)^n\right) \\
&= (2\sin\pi\omega^n)^2 \, (-1)^{(F_n-1)(F_{n-1}+1)} \prod_{r=1}^{F_n-1} 4\left(\sin^2\pi(r\omega + \frac{1}{2}(-\omega)^n) - \sin^2\frac{1}{2}\pi\omega^n\right) \quad (4.16)
\end{aligned}
$$

Now if F_n is odd then $F_n - 1$ is even, and if F_n is even then by (4.5) $F_{n-1} + 1$ is even, and so for any n we have $(-1)^{(F_n-1)(F_{n-1}+1)} = 1$. We have therefore shown that

$$
Q_n^2 = (2\sin\pi\omega^n)^2 \prod_{r=1}^{F_n-1} 4\left(\sin^2\pi(r\omega + \frac{1}{2}(-\omega)^n) - \sin^2\frac{1}{2}\pi\omega^n\right) \quad (4.17)
$$

This completes the re-basing step. We are now ready to develop the expression for Q_n as a perturbation of the rational sine product $|\prod 2\sin\pi r(F_{n-1}/F_n)|$.

The product in (4.17) is empty for $n = 1, 2$. For $n \geq 3$ we develop the second sine term in (4.17) by substituting the Fibonacci identity and then using (4.7) to obtain

$$
\sin\pi(r\omega + (-\omega)^n/2) = \sin\pi\left(\frac{rF_{n-1}}{F_n} - (-\omega)^n(\frac{r}{F_n} - \frac{1}{2})\right) \quad (4.18)
$$

Substituting the residue $t = [rF_{n-1}]$ in the right hand term and using lemma 3.2.1 we obtain

$$
\sin\pi(r\omega + (-\omega)^n/2) = \pm\sin\pi\left(\frac{t}{F_n} - (-\omega)^n(\left\{\frac{(-1)^n F_{n-1}t}{F_n}\right\} - \frac{1}{2})\right) \quad (4.19)
$$

Now observe that $x \mapsto \{x\} - 1/2$ is an odd function (for non-integer x), and we use this fact to simplify the right side to obtain finally for every $1 \leq r \leq F_n - 1$

$$
\left|\sin\pi(r\omega + (-\omega)^n/2)\right| = \left|\sin\pi\left(\frac{t}{F_n} - \omega^n(\left\{\frac{F_{n-1}t}{F_n}\right\} - \frac{1}{2})\right)\right| \quad (4.20)
$$

Now the right hand side is $|s_{nt}|/2$, and for $1 \leq r \leq F_n - 1$ we also have $1 \leq t \leq F_{n-1}$. In this range for t we have $s_{nt} > 0$ by lemma 4.3.3. This gives us for $1 \leq s, t \leq F_n - 1$:

$$
\left|2\sin\pi(r\omega + (-\omega)^n/2)\right| = s_{nt} \quad (4.21)
$$

If we further observe that for $1 \leq r \leq F_n - 1$, $t = [rF_{n-1}]$ runs through a complete set of non-zero residues, so we can rewrite (4.17) for $n \geq 1$ as:

$$\begin{aligned} Q_n^2 &= (2\sin\pi\omega^n)^2 \prod_{t=1}^{F_n-1} \left(s_{nt}^2 - s_{n0}^2\right) \\ &= (2\sin\pi\omega^n)^2 \left(\prod_{t=1}^{F_n-1} s_{nt}\right)^2 \prod_{t=1}^{F_n-1} \left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) \end{aligned} \tag{4.22}$$

We have almost proved lemma 4.4.1. To obtain the final result, we use the standard result that for any p relatively prime to $q \geq 1$

$$\prod_{r=1}^{q-1} 2\sin\left(\frac{\pi r p}{q}\right) = q \tag{4.23}$$

(For a particularly elegant proof see Knill (2012) [31]). From this result we obtain $\prod_{t=1}^{F_n-1} 2\sin\pi\frac{t}{F_n} = F_n$ and the result follows (using $s_{nt} = s_{n(F_n-t)}$ from lemma 4.3.3).

4.5 The Convergence of $C_n = \prod_{t=1}^{(F_n-1)/2}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right)$

In this step we show C_n converges to a strictly positive constant. This is not as straightforward as it appears at first sight as there are terms in s_{nt} which oscillate about 0 but which are not alternating. We therefore cannot assume that C_n is decreasing. Fortunately we are able to compare C_n with a closely related sequence which *is* decreasing and therefore converges.

Theorem 4.5.1. *The sequence $C_n = \prod_{t=1}^{(F_n-1)/2}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right)$ converges to*

$$\prod_{t=1}^{\infty}\left(1 - \frac{1}{20\left(t - \frac{1}{\sqrt{5}}\left(\{t\omega\} - \frac{1}{2}\right)\right)^2}\right) \simeq 0.915$$

For $n \in \{0, 1, 2\}$ the product defining C_n is empty and $C_n = 1$. For the rest of this section we will assume $n \geq 3$, and so by lemma 4.3.3 we have for $1 \leq t \leq F_n - 1$ that $s_{nt} > s_{n0} > 0$. Hence we have $0 < (1 - s_{n0}^2/s_{nt}^2) < 1$ for every term in C_n, and so $0 < C_n < 1$ for $n \geq 3$.

At this point we need to establish some estimates for the terms s_{n0}/s_{nt}. First we develop some general estimates valid for all $0 \leq t < F_n$. For $t = 0$ we have:

$$s_{n0} = 2\sin\pi(\omega^n/2) = \pi\omega^n(1 + O(\omega^{2n})) \tag{4.24}$$

For $1 \leq t \leq F_n/2$, from (4.6) $F_n^{-1} = \sqrt{5}\omega^n(1 + O(\omega^{2n}))$, and from (4.7) $F_{n-1}/F_n = \omega + O(\omega^{2n})$

and so:

$$\begin{aligned} s_{nt} &= 2\sin\pi\left(\left(t\sqrt{5}\omega^n(1+O(\omega^{2n}))\right)-\omega^n\left(\{t\omega\}+tO(\omega^{2n})-\frac{1}{2}\right)\right) \\ &= 2\sin\pi\omega^n t\left(\sqrt{5}-\frac{1}{t}\left(\{t\omega\}-\frac{1}{2}\right)+O(\omega^{2n})\right) \qquad (4.25)\end{aligned}$$

Now let $q=\left\lceil\omega^{-3n/5}\right\rceil$[6]. For $t\geq q$ we use $(\pi/2)\sin x > x$ (from lemma 4.3.1) in (4.25) to give us for large enough n:

$$\begin{aligned} \frac{s_{n0}}{s_{nt}} &< \frac{\pi\omega^n(1+O(\omega^{2n}))}{(2/\pi)2\pi\omega^n t\left(\sqrt{5}-\frac{1}{t}\left(\{t\omega\}-\frac{1}{2}\right)+O(\omega^{2n})\right)} \\ &< \frac{\pi(1+O(\omega^{2n}))}{4q\left(\sqrt{5}-q^{-1}\left(\{t\omega\}-\frac{1}{2}\right)+O(\omega^{2n})\right)} \\ &< \frac{\pi(1+O(q^{-1}))}{4\sqrt{5}q} \\ &= O(q^{-1}) \qquad (4.26)\end{aligned}$$

Now choose $q\leq q_1<q_2\leq F_n/2$. We can now use from lemma 4.3.2 $\prod(1-a_n)>1-\sum|a_n|$ to obtain:

$$\begin{aligned} 1>\prod_{t=q_1}^{q_2}\left(1-\frac{s_{n0}^2}{s_{nt}^2}\right) &\geq \prod_{t=q_1}^{(F_n-1)/2}\left(1-\frac{s_{n0}^2}{s_{nt}^2}\right) \\ &> 1-\sum_{t=q_1}^{(F_n-1)/2}O(q^{-2})>1-F_nO(\omega^{6n/5}) \\ &= 1-O(\omega^{n/5}) \qquad (4.27)\end{aligned}$$

Now we consider the case of $t<q$. From (4.25) we have

$$s_{nt}=2\sin\pi\omega^n t\left(\sqrt{5}-\left(\{t\omega\}-\frac{1}{2}\right)/t+O(\omega^{2n})\right)$$

and the largest term in the argument of the sine function is then $O(\omega^n q)=O(\omega^{2n/5})$, so that for large enough n we can make the argument as small as we like. So we can use $\sin x = x+O(x^3)$ to give us:

$$\begin{aligned} s_{nt} &= 2\pi\omega^n t\left(\sqrt{5}-\frac{1}{t}\left(\{t\omega\}-\frac{1}{2}\right)+O(\omega^{2n})\right)+O(\omega^{6n/5}) \\ &= 2\sqrt{5}\pi\omega^n\left(t-\frac{1}{\sqrt{5}}\left(\{t\omega\}-\frac{1}{2}\right)+O(\omega^{n/5})\right) \qquad (4.28)\end{aligned}$$

[6]Here 3/5 is chosen to optimise convergence, though other values are possible.

We put $u_t = 2\sqrt{5}\left(t - \frac{1}{\sqrt{5}}\left(\{t\omega\} - \frac{1}{2}\right)\right)$. Using (4.24) we get:

$$\frac{s_{n0}}{s_{nt}} = \frac{(1 + O(\omega^{n/5}))}{u_t} \tag{4.29}$$

Hence we can write:

$$\begin{aligned}\prod_{t=1}^{q}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) &= \prod_{t=1}^{q}\left(1 - \frac{1}{u_t^2} - \frac{O(\omega^{n/5})}{u_t^2}\right) \\ &= \prod_{t=1}^{q}\left(1 - \frac{1}{u_t^2}\right)\prod_{t=1}^{q}\left(1 - \frac{O(\omega^{n/5})}{u_t^2 - 1}\right)\end{aligned} \tag{4.30}$$

Now $\sum 1/(u_t^2 - 1)$ converges (by comparison with $\sum 1/t^2 = \pi^2/6$) and so $\sum \frac{O(\omega^{n/5})}{u_t^2-1} = O(\omega^{n/5})$, so by lemma 4.3.2:

$$\prod_{t=1}^{q}\left(1 - \frac{O(\omega^{n/5})}{u_t^2 - 1}\right) = 1 + O(\omega^{n/5}) \tag{4.31}$$

Similarly $\sum 1/u_t^2$ also converges, but for this series we need more information about the limit which we obtain as follows:

$$\begin{aligned}\sum_{t=1}^{\infty}\frac{1}{u_t^2} &< \frac{1}{u_1^2} + \sum_{t=2}^{\infty}\frac{1}{20(t-1)^2} \\ &< 0.056 + \pi^2/120 \\ &< 0.138\end{aligned} \tag{4.32}$$

We now put $U_q = \prod_{t=1}^{q}\left(1 - \frac{1}{u_t^2}\right) > 1 - \sum_{t=1}^{q} 1/u_t^2 > 0.862$. Note that U_q is a descending sequence and bounded below, and so converges to some constant $U_\infty > 0.862$. (In fact we compute $U_\infty \simeq 0.915$). And

$$\begin{aligned}1 > \frac{U_\infty}{U_q} &= \prod_{t=q+1}^{\infty}\left(1 - \frac{1}{u_t^2}\right) > 1 - \frac{1}{20}\sum_{t=q+1}^{\infty}\frac{1}{(t-1)^2} \\ &= 1 - O(q^{-1})\end{aligned}$$

Finally:

$$\begin{aligned}C_n &= \prod_{t=1}^{(F_n-1)/2}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) = \prod_{t=1}^{q}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right)\prod_{t=q+1}^{(F_n-1)/2}\left(1 - \frac{s_{n0}^2}{s_{nt}^2}\right) \\ &= U_\infty\left(1 + O(q^{-1})\right)\left(1 + O(\omega^{n/5})\right)\left(1 - O(\omega^{n/5})\right) \\ &= U_\infty\left(1 + O(\omega^{n/5})\right)\end{aligned}$$

Hence

$$\lim_{n\to\infty} C_n = U_\infty = \prod_{t=1}^{\infty}\left(1 - \frac{1}{20\left(t - \frac{1}{\sqrt{5}}\left(\{t\omega\} - \frac{1}{2}\right)\right)^2}\right) \simeq 0.915 \tag{4.33}$$

4.6 The Convergence of $B_n = \prod_{t=1}^{F_n-1} s_{nt}/\,(2\sin\pi t/F_n)$

In this step we show that B_n converges to a strictly positive limit. In the last section we saw that the proof of convergence was complicated by the presence of non-alternating oscillations in sign. We were able to circumvent this problem by relating the product to one which converged absolutely, and involved a product of square terms $\prod(1 - 1/r^2)$. In this section we are unable to do this as the absolute product behaves like $\prod(1 + 1/r)$ and diverges. The convergence is therefore conditional and we are forced to estimate the compound effects of the signed differences.

Theorem 4.6.1. *The sequence* $\log B_n$ *converges to a finite limit, and the sequence* B_n *to a strictly positive limit.*

We start by examining each term for $1 \le t \le F_n - 1$:

$$\begin{aligned}
\frac{s_{nt}}{2\sin\pi t/F_n} &= \frac{2\sin\pi(t/F_n - \omega^n{}_{nt})}{2\sin\pi t/F_n} \\
&= \cos\pi\omega^n{}_{nt} - \cot\frac{\pi t}{F_n}\sin\pi\omega^n{}_{nt} \\
&= 1 - 2\sin^2\frac{\pi}{2}\omega^n\xi_{nt} - \cot\frac{\pi t}{F_n}\sin\pi\omega^n\xi_{nt}
\end{aligned} \tag{4.34}$$

Put $\alpha_{nt} = 2\sin^2\left(\frac{1}{2}\pi\omega^n\xi_{nt}\right)$ and $h_{nt} = \cot\left(\pi\frac{t}{F_n}\right)\sin\left(\pi\omega^n\xi_{nt}\right)$ so that $B_n = \prod_{t=1}^{F_n-1}(1-\alpha_{nt}-h_{nt})$. We first need an estimate for h_{nt}. Using $\cot x < 1/x$ in $(0,\pi/2)$ and $|\xi_{nt}| < 1/2$ gives us for any $1 \le t \le F_n/2$:

$$\begin{aligned}
|h_{nt}| &= \cot\left(\frac{\pi t}{F_n}\right)\left|\sin\left(\pi\omega^n\xi_{nt}\right)\right| \\
&< \frac{F_n}{\pi t}.\pi\omega^n\,|\xi_{nt}| < \frac{1}{2\sqrt{5}t}\left(1-(-1)^n\omega^{2n}\right) < \frac{1}{4t}
\end{aligned} \tag{4.35}$$

We also have, using $\xi_{nt} - \xi_{\infty t} = t(-\omega)^n/F_n$ and putting $h_{\infty t} = \xi_{\infty t}/\left(\sqrt{5}t\right)$:

$$\begin{aligned}
h_{nt} &= \frac{F_n}{\pi t}\left(1 + O\left(\frac{t^2}{F_n^2}\right)\right)\pi\omega^n\xi_{nt}\left(1+O\left(\omega^{2n}\right)\right) \\
&= \frac{F_n\omega^n}{t}\left(1 - O\left(t^2\omega^{2n}\right)\right)\left(\xi_{\infty t} + t(-\omega)^n/F_n\right) \\
&= h_{\infty t} + O(t\omega^{2n})
\end{aligned} \tag{4.36}$$

We are now in a position to begin our analysis of B_n. Since $|\xi_{nt}| < 1/2$, we have $0 < \alpha_{nt} < \pi^2\omega^{2n}/8$.

Consequently $\log(1-\alpha_{nt}-h_{nt}) = \log(1-h_{nt}) + O(\omega^{2n})$ and we can sum over t to obtain:

$$\left|\log B_n - \sum_{t=1}^{F_n-1} \log(1-h_{nt})\right| = O(\omega^n) \tag{4.37}$$

Writing $B_n^* = \prod_{t=1}^{F_n-1}(1-h_{nt})$, this gives us:

$$B_n \sim B_n^* \tag{4.38}$$

We proceed to investigate the product B_n^*. We start by observing:

$$\log B_n^* = \sum_{t=1}^{F_n-1} \log(1-h_{nt}) = -\sum_{t=1}^{F_n-1}\sum_{k=1}^{\infty}\frac{1}{k}h_{nt}^k \tag{4.39}$$

Using (from lemma 4.3.3) the symmetry $h_{nt} = h_{n(F_n-t)}$, we obtain:

$$\log B_n^* = -2\sum_{t=1}^{(F_n-1)/2}\sum_{k=1}^{\infty}\frac{1}{k}h_{nt}^k = -2\left(\sum_{t=1}^{(F_n-1)/2} h_{nt} + \sum_{t=1}^{(F_n-1)/2}\sum_{k=2}^{\infty}\frac{1}{k}h_{nt}^k\right) \tag{4.40}$$

4.6.1 Convergence of $\sum_{t=1}^{(F_n-1)/2}\sum_{k=2}^{\infty}\frac{1}{k}h_{nt}^k$

It is easy to show that the sum $\sum_{t=1}^{(F_n-1)/2}\sum_{k=2}^{\infty}\frac{1}{k}h_{nt}^k$ is bounded, but it requires rather more work to show that it converges. We start by showing that the contributions from larger values of t, k become negligible. First from (4.35) we have $|h_{nt}| < 1/4t \le 1/4$, and so for $q \ge 2$:

$$\left|\sum_{k=q}^{\infty}\frac{1}{k}h_{nt}^k\right| < \sum_{k=q}^{\infty}\left|h_{nt}^k\right| < \frac{\left|h_{nt}^q\right|}{1-|h_{nt}|} < \frac{4}{3}\left(\frac{1}{4t}\right)^q \tag{4.41}$$

Now put $q = \left\lfloor \omega^{-n/2}\right\rfloor$ and consider values of n large enough that $2 \le q+1 < (F_n-1)/2$. Then:

$$\left|\sum_{t=q+1}^{(F_n-1)/2}\sum_{k=2}^{\infty}\frac{1}{k}h_{nt}^k\right| < \sum_{t=q+1}^{(F_n-1)/2}\frac{4}{3}\left(\frac{1}{4t}\right)^2 < \frac{1}{12}\sum_{t=q+1}^{\infty}\frac{1}{t^2} < \frac{1}{12q} \tag{4.42}$$

$$\left|\sum_{t=1}^{q}\sum_{k=q+1}^{\infty}\frac{1}{k}h_{nt}^k\right| < \sum_{t=1}^{q}\frac{4}{3}\left(\frac{1}{4t}\right)^{q+1} < \frac{1}{3}\left(\frac{1}{4}\right)^q\sum_{t=1}^{\infty}\frac{1}{t^2} < \frac{\pi^2}{18}\left(\frac{1}{4}\right)^q \tag{4.43}$$

Hence both of these sums tend to 0 as $n \to \infty$, and so $\sum_{t=1}^{(F_n-1)/2}\sum_{k=2}^{\infty}\frac{1}{k}h_{nt}^k \sim \sum_{t=1}^{q}\sum_{k=2}^{q}\frac{1}{k}h_{nt}^k$. Note that in the right hand sum $kt \le q^2 < \omega^{-n}$ so that $kt\omega^{2n} < \omega^n \to 0$ with n. We now use (4.36), noting $h_{\infty t} < 1/(2\sqrt{5})$, and taking n large enough that $kt\omega^{2n} \ll 1$, we obtain:

$$\begin{aligned} h_{nt}^k - h_{\infty t}^k &= \left(h_{\infty t} + O(t\omega^{2n})\right)^k - h_{\infty t}^k \\ &= O(kt\omega^{2n}) \end{aligned}$$

It follows that

$$\sum_{t=1}^{q}\sum_{k=2}^{q}\frac{1}{k}(h_{nt}^k - h_{\infty t}^k) = \sum_{k=2}^{q}\sum_{t=1}^{q} O(t\omega^{2n}) = O(q)O(q^2\omega^{2n}) = O(\omega^{n/2})$$

Hence $\sum_{t=1}^{(F_n-1)/2}\sum_{k=2}^{\infty}\frac{1}{k}h_{nt}^k \sim \sum_{t=1}^{q}\sum_{k=2}^{q}\frac{1}{k}h_{nt}^k \sim \sum_{t=1}^{q}\sum_{k=2}^{q}\frac{1}{k}h_{\infty t}^k$. But now we can reuse the arguments of (4.41)–(4.43) to obtain $\sum_{t=1}^{q}\sum_{k=2}^{q}\frac{1}{k}h_{\infty t}^k \sim \sum_{t=1}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k}h_{\infty t}^k$. Finally , using $|h_{\infty t}| < 1/(2\sqrt{5}t)$ we obtain the bound:

$$\sum_{t=1}^{\infty}\sum_{k=2}^{\infty}\left|\frac{1}{k}h_{\infty t}^k\right| < \sum_{t=1}^{\infty}\frac{h_{\infty t}^2}{1-|h_{\infty t}|} < \left(\frac{1}{1-\frac{1}{2\sqrt{5}}}\right)\left(\frac{1}{2\sqrt{5}}\right)^2\frac{\pi^2}{6}$$

Hence the sum above is absolutely convergent, and hence convergent to a limit we denote L_2^B, giving us finally:

$$\lim_{n\to\infty}\sum_{t=1}^{(F_n-1)/2}\sum_{k=2}^{\infty}\frac{1}{k}h_{nt}^k = L_2^B \tag{4.44}$$

4.6.2 Convergence of $\sum_{t=1}^{(F_n-1)/2} h_{nt}$

We are left in (4.40) with estimating the first sum $\sum_{t=1}^{(F_n-1)/2} h_{nt}$. Our estimate of $|h_{nt}| < 1/4t$ is not good enough to help us here as its sum is the (divergent) harmonic series.

Put $H_n = \sum_{t=1}^{(F_n-1)/2} h_{nt} = \sum_{t=1}^{(F_n-1)/2}\cot\left(\frac{\pi t}{F_n}\right)\sin\left(\pi\omega^n\xi_{nt}\right)$ and $H_n^* = \sum_{t=1}^{(F_n-1)/2}\cot\left(\frac{\pi t}{F_n}\right)\sin\left(\pi\omega^n\xi_{\infty t}\right)$ (where in H_n^*we have simply replaced ξ_{nt} with $\xi_{\infty t}$). Note that for $1 \le t \le F_n - 1$ we have $\xi_{nt} - \xi_{\infty t} = t(-\omega)^n/F_n$ so that

$$H_n - H_n^* = \sum_{t=1}^{(F_n-1)/2}\cot\left(\frac{\pi t}{F_n}\right)\pi\omega^n\frac{t(-\omega)^n}{F_n}\left(1+O(\omega^{2n})\right) \tag{4.45}$$

Again for $x \in (0,\pi/2]$ we have $\cot x < 1/x$

$$\begin{aligned}|H_n - H_n^*| &< \sum_{t=1}^{(F_n-1)/2}\left(\frac{\pi t}{F_n}\right)^{-1}\pi\omega^n\frac{t(\omega)^n}{F_n}\left(1+O(\omega^{2n})\right) = \omega^{2n}\sum_{t=1}^{(F_n-1)/2}\left(1+O(\omega^{2n})\right)\\ &= \frac{\omega^n}{2\sqrt{5}} + O(\omega^{3n}) \end{aligned}\tag{4.46}$$

so that $H_n \sim H_n^*$. We now focus on H_n^*. For the next step we will need to revert to summation using integer limits. To do this note that if F_n is even then $h_{F_n/2} = \cot\pi/2\,\sin\pi\,(\omega^n\xi_{F_n/2}) = 0$ so we can ignore this term. So now we can put $M_n = \lfloor (F_n-1)/2\rfloor$ and use summation by parts to obtain:

$$\begin{aligned} H_n^* &= \sum_{t=1}^{M_n} \cot\left(\frac{\pi t}{F_n}\right) \sin\pi\left(\omega^n \xi_{\infty t}\right) \\ &= \sum_{t=1}^{M_n-1} \left(\cot\left(\frac{\pi t}{F_n}\right) - \cot\left(\frac{\pi(t+1)}{F_n}\right)\right) \sum_{s=1}^{t} \sin\pi\left(\omega^n \xi_{\infty s}\right) + \cot\left(\frac{\pi M_n}{F_n}\right) \sum_{s=1}^{M_n} \sin\pi\left(\omega^n \xi_{\infty s}\right) \end{aligned} \tag{4.47}$$

Recalling $|\xi_{\infty t}| < 1/2$, the trailing term is easily estimated as:

$$\left|\cot\left(\frac{\pi M_n}{F_n}\right) \sum_{s=1}^{M_n} \sin\pi\left(\omega^n \xi_{\infty s}\right)\right| < \left(\frac{\pi}{2F_n}\right) \frac{F_n}{2} \left(\frac{\pi\omega^n}{2}\right) + O(\omega^{3n}) = O(\omega^n) \tag{4.48}$$

We can now take limits on (4.47) to obtain, writing $\phi = \pi/F_n$, $C_{nt} = \cot t\phi - \cot(t+1)\phi$ and $S_{nt} = \sum_{s=1}^{t} \sin\pi\left(\omega^n \xi_{\infty s}\right)$:

$$H_n \sim H_n^* \sim \sum_{t=1}^{M_n-1} C_{nt} S_{nt} \tag{4.49}$$

4.6.2.1 The order of the cotangent difference

We estimate the cotangent difference as follows:

$$\begin{aligned} 0 < C_{nt} = \cot t\phi - \cot(t+1)\phi &= \frac{\sin(t+1)\phi\cos t\phi - \cos(t+1)\phi\sin t\phi}{\sin t\phi\sin(t+1)\phi} \\ &= \frac{\sin\phi}{\sin t\phi\sin(t+1)\phi} \end{aligned} \tag{4.50}$$

Substituting back $\phi = \pi/F_n$, and noting from lemma 4.3.1 $(\pi/2)\sin x \geq x$ for $x \in [0, \pi/2]$ we get for $0 < (t+1)\phi \leq \pi/2$, or $1 \leq t \leq (F_n/2) - 1 = M_n$, that $(1/\sin t\phi) \leq F_n/2t$ and hence:

$$\begin{aligned} 0 < C_{nt} &\leq \frac{(\pi/F_n)F_n^2}{4t(t+1)} \\ &< \frac{\pi F_n}{4t^2} \end{aligned} \tag{4.51}$$

For $t < F_n/\pi$ we can be more precise, using $\sin x = x(1 + O(x^2))$ to obtain:

$$\begin{aligned} C_{nt} &= \frac{\phi(1 + O(\phi^2))}{t\phi(1 + O(t^2\phi^2))(t+1)\phi(1 + O(t^2\phi^2))} \\ &= \frac{F_n}{\pi t(t+1)}(1 + O(t^2/F_n^2)) \end{aligned} \tag{4.52}$$

Hence, for $t < F_n/\pi$:

$$\begin{aligned} C_{nt}S_{nt} &= \left(\frac{F_n}{\pi t(t+1)}(1 + O(t^2/F_n^2))\right)\left(\sum_{s=1}^{t} \pi\omega^n \xi_{\infty s}(1 + O(\omega^{2n}))\right) \\ &= \frac{1 + O(t^2\omega^{2n})}{\sqrt{5}t(t+1)} \sum_{s=1}^{t} \xi_{\infty s} \end{aligned} \tag{4.53}$$

4.6.2.2 The order of the partial sums $S_{nt} = \sum_{s=1}^{t} \sin \pi (\omega^n \xi_{\infty s})$

In this step we establish an estimate for S_{nt} in terms of t and n. We will introduce a generalised $S_{nt}(\theta)$ in order to accommodate a dependency on a starting phase angle θ. Our basic approach will then be to find an estimate for $S_{nF_k}(\theta)$ and then express $S_{nt}(0) = S_{nt}$ as a sum of terms involving $S_{nF_k}(\theta)$.

Recall from section 2.5.1 that we can represent $t \geq 1$ as an Ostrowski sum $t = \sum_{s=1}^{m} b_s F_s$ where $m = m(t)$ is the largest integer such that $F_m \leq t$. (This is an approach which has been used by several researchers, eg Knill [31]). Define $t_m = 0$, and for $0 \leq s \leq m-1$ define $t_s = t_{s+1} + b_{s+1}F_{s+1} = \sum_{u=s+1}^{m} b_u F_u$ so that $t_0 = t$.

For $1 \leq r \leq F_n - 1$ we now introduce a generalised $\xi_{\infty r}(\theta) = \{\theta + r\omega\} - 1/2$ so that our $\xi_{\infty t}$ of the previous section is now represented by $\xi_{\infty t}(0)$. We can now use the Ostrowski representation of t to split the sum S_{nt} into segments of length $b_s F_s$:

$$
\begin{aligned}
S_{nt} &= \sum_{r=1}^{t} \sin \pi\omega^n \xi_{\infty r}(0) \\
&= \sum_{r=1}^{b_m F_m} \sin \pi\omega^n \xi_{\infty r}(0) \\
&\quad + \sum_{r=1}^{b_{m-1}F_{m-1}} \sin \pi\omega^n \xi_{\infty r}(b_m F_m \omega) \\
&\quad + \sum_{r=1}^{b_{m-2}F_{m-2}} \sin \pi\omega^n \xi_{\infty r}((b_m F_m + b_{m-1}F_{m-1})\omega) + \dots \\
&= \sum_{s=1}^{m} \sum_{r=1}^{b_s F_s} \sin \pi\omega^n \xi_{\infty r}(t_s \omega) \qquad (4.54)
\end{aligned}
$$

We now introduce a generalised $S_{nt}(\theta) = \sum_{r=1}^{t} \sin \pi (\omega^n \xi_{\infty r}(\theta))$ which allows us to write for $1 \leq t \leq F_n - 1$:

$$
S_{nt} = S_{nt}(0) = \sum_{s=1}^{m} b_s S_{nF_s}(t_s \omega) \qquad (4.55)
$$

We proceed to study the order of the terms $S_{nF_s}(\theta)$. Now F_{i-1}/F_i is a convergent to ω, so from (2.7) we have $\left|\sum_{p=1}^{F_i} \left(\{\theta + p\omega\} - \frac{1}{2}\right)\right| < 3/2$. We fix n, and use the result to estimate S_{nF_i} for $1 \leq i < n$.

Using $\sin x = x + O(x^3)$ and, from (4.6), $F_i\omega^{2n} \leq F_n\omega^{2n} = O(\omega^n)$, we obtain:

$$
\begin{aligned}
|S_{nF_i}(\theta)| &= \left|\sum_{p=1}^{F_i} \sin \pi\omega^n \left(\{\theta + p\omega\} - \frac{1}{2}\right)\right| \\
&= \pi\omega^n \left|\sum_{p=1}^{F_i} \left(\{\theta + p\omega\} - \frac{1}{2} + O(\omega^{2n})\right)\right| \\
&< \pi\omega^n \left(\frac{3}{2} + O(\omega^n)\right) \qquad (4.56)
\end{aligned}
$$

Note that the calculation of the $O(\omega^n)$ term is independent of the value of i for $i \leq n$. We are now in a position to estimate S_{nt} for $1 \leq t \leq F_n - 1$, using (4.55) for $1 \leq t \leq F_n - 1$:

$$\begin{aligned} |S_{nt}(0)| &= \left|\sum_{s=1}^{m} b_s S_{nF_s}(t_s\omega)\right| < \sum_{s=1}^{m} b_s \pi\omega^n \left(\frac{3}{2} + O(\omega^n)\right) \\ &< \pi\omega^n \left(\frac{3}{2} + O(\omega^n)\right) \sum_{s=1}^{m} b_s \end{aligned} \tag{4.57}$$

We now need an estimate of m which we develop in the following lemma:

Lemma 4.6.2. *If $n \geq 1$ has the Ostrowski representation $\sum_{s=1}^{m} b_s F_s$, then the representation does not contain any consecutive Fibonacci numbers, and:*

$$m \leq \lfloor (\log n + 1) / \log(1+\omega) \rfloor \tag{4.58}$$

$$\sum_{s=1}^{m} b_s \leq \lfloor (\log n + 1) / \log(2+\omega) \rfloor \tag{4.59}$$

Proof. By definition m is the index of the the Fibonacci number satisfying $F_m \leq n < F_{m+1}$. Using the identity (4.6) $F_m = (\omega^{-m} - (-\omega)^m)/\sqrt{5}$ we deduce (using $\log(1+x) < x$, and $\omega^{-1} = 1 + \omega$):

$$\begin{aligned} m &= \max\{j : \omega^{-j} \leq \sqrt{5}n + (-\omega)^j\} \\ &= \max\{j : j \leq \log\left(\sqrt{5}n + (-\omega)^j\right) / \log(1+\omega) \\ &\leq \lfloor (\log n + 1) / \log(1+\omega) \rfloor \end{aligned} \tag{4.60}$$

Now suppose the representation has $b_i = 1$. This means that at some step the Ostrowski algorithm has processed some integer k lying in $F_i \leq k < F_{i+1}$. But then the next step would process $k - F_i$, and since $F_{i-1} + F_i = F_{i+1}$ the constraint $k < F_{i+1}$ gives $k - F_i < F_{i-1}$ so that $b_{i-1} = 0$. Hence the representation cannot include two consecutive Fibonacci numbers. But since $F_2 = F_1 = 1$ we also always have $b_1 = 0$. This gives us:

$$\sum_{s=1}^{m} b_s \leq \lfloor m/2 \rfloor \tag{4.61}$$

The result follows using $(1+\omega)^2 = 2 + \omega$. □

Substituting this result in (4.57) means we have established:

Lemma 4.6.3. *For $1 \leq t \leq F_n - 1$, the partial sums $S_{nt} = \sum_{s=1}^{t} \sin \pi (\omega^n \xi_{\infty t})$ satisfy:*

$$|S_{nt}| < \frac{3}{2}\pi\omega^n \lfloor (\log t + 1) / \log(2+\omega) \rfloor + O(n\omega^{2n}) \tag{4.62}$$

In particular we can find a K independent of n such that $|S_{nt}| < K\omega^n(\log t + 1)$

4.6.2.3 Conclusion of proof of convergence of the first sum

From lemma 4.6.3 in section 4.6.2.2 we have $|S_{nt}| = \left|\sum_{s=1}^{t} \sin\pi\left(\omega^n \xi_{\infty s}\right)\right| < K\omega^n(\log t + 1)$ for some K independent of n. Combining this with (4.51) we get $|C_{nt}S_{nt}| < K(1+\omega^{2n})(\log t + 1)/\sqrt{5}t^2$. But $\sum_{t=1}^{\infty}(\log t + 1)/t^2$ is absolutely convergent, so putting $K' = (1+\omega^2)K\sum_1^{\infty}(\log t + 1)/\sqrt{5}t^2$ we get:

$$\sum_{t=1}^{M_n-1} |C_{nt}S_{nt}| < K'$$

Now put $q = \left\lfloor \omega^{-n/2} \right\rfloor$, so using (4.53) and for n large enough that $q < M_n - 1$ we have:

$$\begin{aligned} K' &> \sum_{t=1}^{q} |C_{nt}S_{nt}| = \sum_{t=1}^{q}\left|\frac{1+O(t^2\omega^{2n})}{\sqrt{5}t(t+1)}\sum_{s=1}^{t}\xi_{\infty s}\right| \\ &= \sum_{t=1}^{q}\left(\left|\frac{1}{\sqrt{5}t(t+1)}\sum_{s=1}^{t}\xi_{\infty s}\right|(1+O(\omega^n))\right) \end{aligned} \tag{4.63}$$

This tells us that the series $\sum_{t=1}^{q}\left(\frac{1}{\sqrt{5}t(t+1)}\sum_{s=1}^{t}\xi_{\infty s}\right)$ is absolutely convergent, and hence converges to a limit L_1^B, and hence so does the sequence whose n-th term is $\sum_{t=1}^{q}|C_{nt}S_{nt}|$. But noting that $\sum_{t=q+1}^{\infty}(\log t + 1)/t^2$ is of order $O(\log q/q)$, we can deduce that

$$\lim_{n\to\infty}\sum_{t=1}^{M_n-1} C_{nt}S_{nt} = \lim_{n\to\infty}\left(\left(\sum_{t=1}^{q} C_{nt}S_{nt}\right) + O(\log q/q)\right) = L_1^B$$

From (4.49) this gives us:

$$H_n \sim H_n^* \sim \sum_{t=1}^{M_n-1} C_{nt}S_{nt} \longrightarrow L_1^B \tag{4.64}$$

4.6.3 Conclusion of proof of convergence of B_n

Combining (4.38), (4.40), (4.44) and (4.64) gives us finally

$$\log B_n \to -2\left(L_1^B + L_2^B\right) \tag{4.65}$$

and noting that both limits are finite establishes theorem 4.6.1.

4.7 Two additional results

In this section we show how the other two results of theorem 4.1.2 flow from our main result $p_{F_n}(\omega) \to c$. The first result is really just a direct corollary of our main result.

4.7.1 The convergence of $P_{F_n-1}(\omega)/F_n$

Corollary 4.7.1. *The sequence $P_{F_n-1}(\omega)/F_n$ converges to $c\sqrt{5}/2\pi$ where c is the limit of the sequence $P_{F_n}(\omega)$*

Proof. Since $P_{F_n-1}(\omega) = P_{F_n}(\omega)/2\sin\pi\omega^n \sim c/2\pi\omega^n$, the result follows from $F_n \sim \omega^{-n}/\sqrt{5}$ □

4.7.2 The power law growth of $P_k(\omega)$ for general k

We now turn to the more important result that the growth and decay of $P_k(\omega)$ are bounded by power laws, specifically:

Theorem 4.7.2. *There are real constants $E_1 \le 0 < 1 \le E_2$ independent of k such that for $k \ge 2$ we have $k^{E_1} \le P_k(\omega) \le k^{E_2}$*

The main part of the proof is to establish that these constants exist. If they do then our main result ($P_{F_n}(\omega) \longrightarrow c$) shows we must have $E_1 \le 0$, and Proposition 4.7.1 ($P_{F_n-1}(\omega)/F_n \longrightarrow c\sqrt{5}/2\pi$) shows we must have $E_2 \ge 1$.

Knill and Tangerman provide an outline proof of existence in the logarithmic case, but appear to make an assumption which, although correct, seems to us to require its own proof. We will give the outline proof here, and then complete it rigorously. We begin by using the techniques of section 4.6.2.2 to split the product $P_k(\omega)$ into sub-products whose lengths are Fibonacci numbers.

Recall from section (2.5.1) that we can express any integer $k \ge 0$ as a sum of Fibonacci numbers $\sum_{s=1}^{m} b_s F_s$. For $0 \le s \le m-1$ put $k_s = \sum_{u=s+1}^{m} b_u F_u$, $k_m = 0$ so that for $m > 1$ we can split the overall product into sub-products of length $b_s F_s$ (regarding the empty product as 1) to get:

$$
\begin{aligned}
P_k(\omega) &= \prod_{r=1}^{k} |2\sin\pi(r\omega)| \\
&= \prod_{r=1}^{b_m F_m} |2\sin\pi(r\omega)| \\
&\quad \times \prod_{r=1}^{b_{m-1}F_{m-1}} |2\sin\pi(r\omega + b_m F_m\omega)| \\
&\quad \times \prod_{r=1}^{b_{m-2}F_{m-2}} |2\sin\pi(r\omega + (b_m F_m + b_{m-1}F_{m-1})\omega)| \times \ldots \\
&= \prod_{s=1}^{m} \prod_{r=1}^{b_s F_s} |2\sin\pi(r\omega + k_s\omega)| \qquad (4.66)
\end{aligned}
$$

Now since $b_m = 1$, the term for $s = m$ of this product is $\prod_{r=1}^{F_m} |2\sin\pi(r\omega)| = P_{F_m}(\omega) \sim c$ (by the main result of this chapter), and it is also strictly positive, so that we can find constants $0 < C_1 < C_2$ bounding $P_{F_m}(\omega)$ for all m. We will claim here and prove later that we can do the same for the other terms:

Claim 4.7.3. Assume that we can choose real constants C_1, C_2 with $0 < C_1 < C_2$ such that they bound all the terms in (4.66), ie so that for each $1 \le s \le m$:

$$C_1 \le \prod_{r=1}^{b_s F_s} |2\sin\pi(r\omega + k_s\omega)| \le C_2 \tag{4.67}$$

(Note that in order to bound empty products this requires $C_1 \le 1 \le C_2$).

Then we have from (4.66):

$$C_1^m \le P_k(\omega) \le C_2^m \tag{4.68}$$

Now for $m \ge 1$, using (4.6), we obtain $\log k \ge \log F_m = \log\left(\omega^{-m}\left(1-(-1)^m\omega^{2m}\right)/\sqrt{5}\right) > m\log\omega^{-1} + \log((1-\omega^4)/\sqrt{5})$ which gives $m < \left(\log(k\sqrt{5}/(1-\omega^4))\right)/\log\omega^{-1}$. Substituting in (4.68), and using $C_1 \le 1 \le C_2$ gives us:

$$\left(\frac{k\sqrt{5}}{1-\omega^4}\right)^{\log C_1} < P_k(\omega) < \left(\frac{k\sqrt{5}}{1-\omega^4}\right)^{\log C_2} \tag{4.69}$$

from which theorem 4.7.2 follows easily.

This is essentially an amplified version of the outline proof provided by Knill and Tangerman, although we have provided it in multiplicative form, rather than the additive (logarithmic) form used in the aforementioned paper. The assumption in claim 4.7.3 is in fact correct (and it is trivial if $b_s = 0$), but a proof does not appear trivial for $b_s = 1$, and so we provide one here.

Since the case $b_s = 0$ is trivial we need to deal only with $b_s = 1$. By the rules of the Fibonacci representation (see lemma 4.6.2), $b_r = 1$ implies $b_{r+1} = 0$. Hence $k_s\omega = \sum_{u=s+2}^m b_u F_u \omega = N + \sum_{u=s+2}^m -b_u(-\omega)^u$ for some integer N. Now $\left|\sum_{u=s+2}^m -b_u(-\omega)^u\right| \le \omega^{s+2}(1+\omega^2+\omega^4\ldots) < \omega^{s+1}$. Hence the claim is proved if we can prove the slightly more general assertion:

Lemma 4.7.4. *There are real constants C_1, C_2 satisfying $0 < C_1 \le 1 \le C_2$ such that $C_1 \le \prod_1^{F_n} |2\sin\pi(r\omega + y)| \le C_2$ whenever $n \ge 2$ and $|y| \le \omega^{n+1}$.*

Note the lemma does not hold for $n = 1$ as $\prod_1^{F_n} |2\sin\pi(r\omega + y)| = 0$ for $y = \omega^2$.

We begin by expanding the sine product as follows:

$$\prod_1^{F_n} |2\sin\pi(r\omega + y)| = \prod_1^{F_n} |2\sin\pi(r\omega)|\,|\cos\pi y + \cot\pi r\omega.\sin\pi y| \tag{4.70}$$

For $n \ge 2$ and $|y| \le \omega^{n+1}$ it is easy to calculate that $\cos\pi y + \cot\pi r\omega.\sin\pi y > 0$. We can therefore take logs of the product above to obtain:

$$\log\prod_1^{F_n}|2\sin\pi(r\omega+y)| = \log P^{Sud}_{F_n}(\omega) + \sum_1^{F_n}\log\left(1-2\sin^2\frac{\pi y}{2}+\cot\pi r\omega.\sin\pi y\right) \qquad (4.71)$$

Since $P_{F_n}(\omega)$ is already suitably bounded by the main result of this chapter, it remains to show that the log-sum is bounded above and below. We begin with establishing the upper bound as this is slightly more straightforward than the lower bound.

4.7.2.1 The upper bound on the growth rate

We use $\log(1+x)\le x$ for $x\in(-1,1]$ to obtain:

$$\begin{aligned}\sum_{r=1}^{F_n}\log\left(1-2\sin^2\frac{\pi y}{2}+\cot\pi r\omega.\sin\pi y\right) &< \sum_{r=1}^{F_n}\left(-2\sin^2\frac{\pi y}{2}+\cot\pi r\omega.\sin\pi y\right)\\ &= -2F_n\sin^2\frac{\pi y}{2}+\sin\pi y\sum_{r=1}^{F_n}\cot\pi r\omega\\ &< \pi|y|\left|\sum_{r=1}^{F_n}\cot\pi r\omega\right| \qquad (4.72)\end{aligned}$$

Now from (3.8) $\left|\sum_{r=1}^{F_n}\cot\pi r\omega\right| < \frac{1}{\pi\omega^n}+\frac{\pi\omega^n}{2}$.From (4.72), we therefore obtain for $n\ge 2$:

$$\sum_{r=1}^{F_n}\log\left(1-2\sin^2\frac{\pi y}{2}+\cot\pi r\omega.\sin\pi y\right) \le \pi|y|\left(\frac{1}{\pi\omega^n}+\frac{\pi\omega^n}{2}\right)\le|y|\left(\omega^{-n}+\frac{\pi^2\omega^n}{2}\right)$$

This establishes the upper bound we needed, and also in (4.70) we now have for $n\ge 2$ and $|y|\le\omega^{n+1}$:

$$\prod_1^{F_n}|2\sin\pi(r\omega+y)|\le P_{F_n}(\omega)\exp\left(|y|\left(\omega^{-n}+\frac{\pi^2\omega^n}{2}\right)\right) \qquad (4.73)$$

This now also establishes the upper bound in (4.68) and hence also in (4.69). We now turn to the lower bound.

4.7.2.2 The lower bound on the growth rate

For the upper bound we were able to use the standard result that $\log(1+x)\le x$ for $-1<x\le 1$. We now need a lower bound for the logarithm. The following lemma provides this:

Lemma 4.7.5. *For real* $1\ge x>-0.683$ *we have* $x\ge\log(1+x)>x-x^2$

Proof. For $x>-1$ put $f(x)=\log(1+x)-(x-x^2)$. Note the function is continuous on $(-1,\infty)$ and that $f(0)=0$. It is easy to verify that this has critical points at $x=0,-0.5$ and the derivative is positive on $(0,\infty)$ and negative on $(-0.5,0)$ so that the function itself is positive on these two intervals. On $(-1,-0.5)$ the derivative is negative so the function descends with descending x from its maximum at $x = -0.5$ to a zero in $(-1, -0.5)$. A numerical calculation shows the root lies just below $x = -0.683$. □

We wish to apply the lemma to the expression $\sum_{r=1}^{F_n} \log\left(1 - 2\sin^2\frac{\pi y}{2} + \cot\pi r\omega.\sin\pi y\right)$ from (4.72). To do this we must first establish that $-2\sin^2\frac{\pi y}{2} + \cot\pi r\omega.\sin\pi y > -0.683$. Now for $n \geq 4$ and $|y| < \omega^{n+1}$ we have:

$$\begin{aligned} \left|-2\sin^2\frac{\pi y}{2} + \cot\pi r\omega.\sin\pi y\right| &< 2\frac{\pi^2 y^2}{4} + \pi y\cot\pi\omega^n \\ &< \frac{\pi^2\omega^{2n+2}}{2} + \pi\omega^{n+1}\frac{1}{\pi\omega^n(1-\pi^2\omega^{2n}/6)} \\ &< \frac{\pi^2\omega^{10}}{2} + \frac{\omega}{(1-\pi^2\omega^8/6)} \\ &< 0.681 \end{aligned}$$

We can now apply lemma 4.7.5 to obtain:

$$\begin{aligned} \sum_{r=1}^{F_n}\log\left(1 - 2\sin^2\frac{\pi y}{2} + \cot\pi r\omega.\sin\pi y\right) &\geq \sum_{r=1}^{F_n}\left(-2\sin^2\frac{\pi y}{2} + \cot\pi r\omega.\sin\pi y\right) \\ &\quad - \sum_{r=1}^{F_n}\left(-2\sin^2\frac{\pi y}{2} + \cot\pi r\omega.\sin\pi y\right)^2 \\ &= \sum_{r=1}^{F_n} -2\sin^2\frac{\pi y}{2} - 4\sin^4\frac{\pi y}{2} \\ &\quad + \sum_{r=1}^{F_n}\left(1 + 4\sin^2\frac{\pi y}{2}\right)\cot\pi r\omega.\sin\pi y - (\cot\pi r\omega.\sin\pi y)^2 \\ &\geq F_n\left(-2\left(\frac{\pi\omega^{n+1}}{2}\right)^2 - 4\left(\frac{\pi\omega^{n+1}}{2}\right)^4\right) \\ &\quad - \left(1 + 4\left(\frac{\pi\omega^{n+1}}{2}\right)^2\right)\left|\sum_{r=1}^{F_n}\cot\pi r\omega.\sin\pi y\right| \\ &\quad - \sum_{r=1}^{F_n}(\cot\pi r\omega.\sin\pi y)^2 \end{aligned} \tag{4.74}$$

The first term is clearly bounded below (and converges to 0). From (3.8) and for $n \geq 2$, we have shown $\left|\sum_{r=1}^{F_n}\cot\pi r\omega.\sin\pi y\right| < \omega + \pi^2\omega^{2n+1}/2$, and so the second term is also bounded below. It remains to show that the third term is bounded below. Using lemma 4.3.1 (and allowing for $y = 0$) we have for $n \geq 1$:

$$\sum_{r=1}^{F_n}(\cot\pi r\omega.\sin\pi y)^2 \leq (\pi y)^2\sum_{r=1}^{F_n}\cot^2\pi r\omega$$

We now use the same argument as in section 3.2. We put $r_p = rF_{n-1} \bmod F_n$,we obtain for $n \geq 3$, $\cot^2\pi r\omega < \cot^2(\pi r_p/F_n)$ for $0 \leq r_p \leq \left\lfloor\frac{1}{2}F_n - 1\right\rfloor$ and for $n \geq 4$ it also gives $\cot^2\pi r\omega <$

$\cot^2(\pi(r_p+1)/F_n)$ for $\left\lceil \frac{1}{2}F_n \right\rceil \le r_p \le F_n - 1$. There is a special case: when $n \ge 4$ and F_n is odd there is an uncovered interval $[\frac{1}{2}(F_n-1)/F_n, \frac{1}{2}(F_n+1)/F_n]$, but here again $\cot^2 \pi r\omega < \cot^2 \pi r_p/F_n$ for $r_p = \frac{1}{2}(F_n-1) = \left\lfloor \frac{1}{2}F_n \right\rfloor$. We are now almost ready to sum over r, but we again need to take care of singularities, and these occur this time at $r_p = 0, F_n - 1$, corresponding to $r = F_n, [(-1)^n F_{n-1}]$. In the second case, either $r = F_{n-1}$ (n even) or $F_n - F_{n-1} = F_{n-2}$ (n odd). The $r = F_{n-1}$ case gives the larger result, and hence for $n \ge 4$, using $|\cot x| < |1/x|$:

$$
\begin{aligned}
\sum_{r=1}^{F_n} \cot^2 \pi r\omega &< \left(\sum_{r_p=1}^{\lfloor \frac{1}{2}F_n \rfloor} \cot^2 \pi \frac{r_p}{F_n} + \sum_{r_p=\lceil \frac{1}{2}F_n \rceil}^{F_n-2} \cot^2 \pi \frac{r_p+1}{F_n} \right) + \cot^2 \pi F_n\omega + \cot^2 \pi F_{n-1}\omega \\
&< 2 \sum_{r_p=1}^{\lfloor \frac{1}{2}F_n \rfloor} \left(\frac{F_n}{\pi r_p} \right)^2 + \left(\frac{1}{\pi\omega^n} \right)^2 + \left(\frac{1}{\pi\omega^{n-1}} \right)^2 \\
&< \frac{2F_n^2}{\pi^2} \left(\frac{\pi^2}{6} \right) + \frac{1+\omega^2}{\pi^2\omega^{2n}} \qquad (4.75)
\end{aligned}
$$

Hence for $n \ge 4$:

$$
\begin{aligned}
\sum_{r=1}^{F_n} (\cot \pi r\omega . \sin \pi y)^2 &< \pi^2\omega^{2n+2} \left(\frac{1}{3} \cdot \frac{1}{5} (\omega^{-n} + \omega^n)^2 + \frac{1+\omega^2}{\pi^2\omega^{2n}} \right) \\
&\le \omega^2 \left(\frac{\pi^2}{15} (1 + 2\omega^8 + \omega^{16}) + (1+\omega^2) \right)
\end{aligned}
$$

Hence the third term in (4.74) is also bounded below, and the lower bound we needed for this log-sum is also established for $n \ge 4$. Hence for $n \ge 4$, (4.70) is bounded below by a strictly positive constant, and in fact it is easily verified that this is also true for $n = 2, 3$, finally establishing lemma 4.7.4.

This now also establishes the lower bound in (4.68) and hence also in (4.69).

Chapter 5

Improved bounds for Sudler's product

5.1 Introduction

In the previous chapter we studied Sudler's product of sines at the golden rotation, given by $P_n(\omega) = \prod_{k=1}^{n} |2\sin\pi(k\omega)|$. Our central result was that $P_{F_n}(\omega) \to c$ for Fibonacci numbers F_n. We used this to provide an alternative proof of the result that the product has power law growth, ie there are $C_1 \leq 0, C_2 \geq 1$ such that $k^{C_1} \leq P_k(\omega) \leq k^{C_2}$ for $k \geq 2$.

In this chapter we will build and improve on these results. In particular:

1. We will sharpen these bounds by showing there is a constant lower bound lying above 1, and a linear upper bound.

2. We will settle (negatively) an open question of Erdős-Szekeres-Lubinsky that for all α we have $\liminf_{n\to\infty} P_n(\alpha) = 0$

The latter question was introduced by Erdős and Szekeres in 1959 ([11]) in which they proved $\liminf_n P_n(\alpha) = 0$ for ae α, and suggested it may hold for all α. Lubinsky improved the result by showing that there is a $K > 0$ such that $\liminf_n P_n(\alpha) = 0$ for all α except those of constant type[1] $< K$ ([41]). He went on to profess he was "certain" that the result would hold for all α, but in fact we shall show that it does not.

[1]See section 2.3.1

5.2 Power law exponents

5.2.1 The role of renormalisation

This far we have not formally introduced the concept of renormalisation into our analyses. However we have been using renormalisation ideas informally, and this is a good point at which to make our use of the tools explicit. Renormalisation has two particular tools: decimation and rescaling. In studying not the whole sequence $S_n(x,\alpha)$, but instead the sequence S_{q_n}, we have been studying a decimation. In the earlier analyses there was no obvious use of rescaling, as we were effectively studying a sequence of constant functions in which rescaling has no effect. The net effect is that we were using a degenerate form of renormalisation, and this was all we required up to and including the proof of $P_{F_n}(\omega) \to c$. However in deducing the power law growth result, we were then forced to consider not just the constant function $P_{F_n}(\omega)$, but the non-constant function $P_{F_n}(\omega + y)$ (see lemma 4.7.4). From this point our results were only valid when $|y| \leq \omega^{n+1}$ for each n. In other words, our results were only valid when we rescaled the domain of y for each n.

In this chapter we will make our use of rescaling explicit by rescaling the variable rather the domain: we will keep the domain of y constant, but use the rescaled value $(-\omega)^n y$ as the argument in our functions[2].

5.2.2 Definitions and notation

We will simplify notation by introducing the product $P_k(y) = P_k(\omega + y) = \prod_1^k |2\sin\pi(r\omega + y)|$, so that $P_k(0) = P_k(\omega)$. A key role will be played by the interval $I = [-\omega, \omega^2]$.

We will be working again with the Fibonacci decimation $P_{F_n}(y)$ of the general sequence $P_k(y)$. Given $x \geq 0$, x is contained in a unique Fibonacci interval, ie there is an $n \geq 0$ such that $x \in [F_n, F_{n+1})$ (or equivalently $F_n \leq x < F_{n+1}$). Then F_n is the largest Fibonacci number $\leq x$, and by analogy with the integer floor function $\lfloor . \rfloor$ we call F_n the *Fibonacci floor* of x and denote it $\lfloor x \rfloor_F$. We call n the *Fibonacci index* of x and denote it $\iota(x)$.

Assume now that x is an integer k. There is an important technical detail in handling low values of k: for $n \geq 4$, there are multiple values of k in the Fibonacci interval $I_n = [F_n, F_{n+1})$, but for $k = 0, 1, 2, 3, 4$ we have $\iota(k) = 0, 2, 3, 4, 4$ and so for $n = 0, 2, 3$ there is a single value of k in I_n, and in the case of $n = 1$, there are no values. We will need to take great care with these special cases in the sequel.

[2] For technical reasons, it proves simpler to use $(-\omega)^n y$ rather than the more obvious $\omega^n y$.

5.2.3 Amplified statement of main results

Recall in chapter 4 we established that the Sudler products contained in in the Fibonacci decimation satisfy $P_{F_n}(\omega) \to c > 0$ and $P_{F_n-1}(\omega) \sim F_n c\sqrt{5}/2\pi$. We have also established (theorem 4.7.2) that the growth of the general element $P_k(\omega)$ is bounded by power laws.

In this chapter we improve this result to show that the upper growth bound is in fact linear, and the lower bound is a constant:

Theorem 5.2.1. *For some constant C independent of $k \geq 1$ we have:*

$$1 < P_{F_n}(\omega) \leq P_k(\omega) < Ck$$

where $F_n = \lfloor k \rfloor_F$ is the Fibonacci floor of k, ie $n = \max\{m : F_m \leq k\}$. Equality occurs only for $k = F_n$.

This immediately gives us:

Corollary 5.2.2. $\liminf_{k\to\infty} P_k(\omega) = \liminf_{k\to\infty} P_{F_n}(\omega) = c > 0$

This settles (negatively) the open question of Erdős-Szekeres-Lubinsky referred to in the introduction (section 5.1).We prove the lower bound of theorem 5.2.1 first (which is the hard part), then use this lower bound to prove the upper bound.

5.3 Proof of the lower bound $P_k(\omega) \geq P_{F_n}(\omega) > 1$

5.3.1 Outline of the proof

The overall logic of the proof is simple but the proof itself is quite lengthy and so we provide an outline here. The proof breaks into three main steps, each of which has its own section below. Recall that we write $P_k(x) := P_k(\omega + x)$, and $P_k = P_k(0)$. The three steps are then:

1. For $y \in I = [-\omega, \omega^2]$ and F_n the Fibonacci floor of $k \geq 0$ we have the lower bound $P_k((-\omega)^n y) \geq P_{F_n}((-\omega)^n y)$ if the Fibonacci Hypothesis 5.3.3 holds, namely for $y \in I$ and $m \geq 2$, $P_{F_m}((-\omega)^m y) > 1$ (see section 5.3.2).

2. The Fibonacci Hypothesis holds if for $m \geq 2$, $P_{F_m} > P$ for a certain constant $P > 1$ (see section 5.3.3).

3. For $m \geq 2$, $P_{F_m} > P$ (see section 5.3.4).

The last step requires the most work, and its proof will occupy a significant part of this chapter.

5.3.2 A lower bound on $P_k((-\omega)^n y)$

We start by establishing that $P_k((-\omega)^n y) \geq P_{F_n}((-\omega)^n y)$ is equivalent to the inequality $P_{k-F_n}((-\omega)^n(y-1)) \geq 1$.

Lemma 5.3.1. *Let F_n be the Fibonacci floor of $k \geq 0$. Then:*

$$P_k((-\omega)^n y) = P_{F_n}((-\omega)^n y) P_{k-F_n}((-\omega)^n(y-1))$$

Proof. Note that $k \geq F_n$ and so using the remark following lemma 2.5.3, for $n \geq 0$, $P_k(x) = P_{F_n}(x)P_{k-F_n}(x+F_n\omega)$. Using $\{F_n\omega\} = \{-(-\omega)^n\}$ and putting $y = (-\omega)^{-n}x$ gives us $P_k((-\omega)^n y) = P_{F_n}((-\omega)^n y)P_{k-F_n}((-\omega)^n y - (-\omega)^n)$, and the result follows immediately. □

Lemma 5.3.2. *Let $y \in I = [-\omega, \omega^2]$. Then $(-\omega)^n(y-1) \in I$ for $n \geq 2$*

Proof. Write $I-1$ to denote the translation of I by -1, so that $I-1 = [-\omega-1, \omega^2-1] = [-\omega^{-1}, -\omega]$ and so for $n \geq 2$ and even, $(-\omega)^n(I-1) = [-\omega^{n-1}, -\omega^{n+1}] \subseteq [-\omega, -\omega^3] \subset I$, and for $n \geq 3$ and odd $(-\omega)^n(I-1) = [\omega^{n+1}, \omega^{n-1}] \subseteq [\omega^4, \omega^2] \subset I$ □

Note that in the proof above the two endpoints are achieved (ie $-\omega = (-\omega)^2(-\omega-1)$, and $\omega^2 = (-\omega)^3(-\omega-1)$) so that I is also the smallest interval with the property of the lemma[3].

We are now almost in a position to prove the main result of this section, namely that $P_k((-\omega)^n y)$ is bounded below by $P_{F_n}((-\omega)^n y)$. The missing piece is proved in the subsequent sections, and we will state it here as a hypothesis:

Definition 5.3.3. Fibonacci Hypothesis (FH): For $y \in I$ and $n \geq 2$, $P_{F_n}((-\omega)^n y) > 1$

Note that for $n = 0$ by definition $P_{F_0}(y) = 1$, but for $n = 1$ we have $P_{F_1}((-\omega)y) = |2\sin\pi(\omega - (-\omega)y)| = 0$ for $y = -\omega$, hence the restriction to $n \geq 2$ above. In addition, as described in section 5.2.2 , there is a small technical issue which we need to take care of, and this is the fact that the Fibonacci floor of $k = 0$ is given by $n = 0$, but the Fibonacci floor of $k = 1$ is $n = 2$ (due to the fact that $F_1 = F_2 = 1$). This is the reason for stating the inductive hypothesis in terms of k in our next lemma, rather than n which might at first sight seem more natural.

Lemma 5.3.4. *Let $\iota(k)$ be the Fibonacci index of k (so that $F_{\iota(k)} \leq k < F_{\iota(k)+1}$), and let $y \in I$. Then from the Fibonacci Hypothesis we can deduce:*

For any $k \geq 0$, $P_k((-\omega)^{\iota(k)}y) \geq P_{F_{\iota(k)}}((-\omega)^{\iota(k)}y) \geq 1$ with first equality only for $k = F_{\iota(k)}$ and second equality only for $k = F_0 = 0$.

[3] Another way to see this is that mapping $y \mapsto (-\omega)^n(y-1)$ for $n \geq 2$ is actually an iterated composition of the maps $y \mapsto \omega^2(y-1)$ and $y \mapsto -\omega y$. Regarding these maps as an IFS we can compute its attractor as $I = [-\omega, \omega^2]$. However we do not need this additional theory here.

Proof. For $y \in I$, let $IH(m)$ be the inductive hypothesis: For $0 \leq k \leq m$, $P_k((-\omega)^{\iota(k)}y) \geq P_{F_{\iota(k)}}((-\omega)^{\iota(k)}y) \geq 1$ with first equality only for $k = F_{\iota(k)}$ and second equality only for $k = F_0 = 0$.

For $k = 0$, $P_k(y) = 1$ for all y and so $IH(0)$ is satisfied trivially, and with equality.

We now suppose $IH(m-1)$ holds for some $m \geq 1$. Note that for $m \geq 1$ we have $\iota(m) \geq 2$. For $m \geq 1$, $IH(m)$ asserts:

$$P_k((-\omega)^{\iota(m)}y) \geq P_{F_{\iota(m)}}((-\omega)^{\iota(m)}y) > 1$$

with equality only for $m = F_{\iota(m)}$.

But by lemma 5.3.1, $P_m((-\omega)^{\iota(m)}y) = P_{F_{\iota(m)}}((-\omega)^{\iota(m)}y)P_{m-F_{\iota(m)}}((-\omega)^{\iota(m)}(y-1))$. Now $(-\omega)^{\iota(m)}(y-1) = (-\omega)^{\iota(m-F_{\iota(m)})}(-\omega)^j(y-1)$ where $j = \iota(m) - \iota(m - F_{\iota(m)})$. But since $\iota(m) \geq 2$ we also have $m - F_{\iota(m)} < F_{\iota(m)-1}$, and so $\iota(m - F_{\iota(m)}) \leq \iota(m) - 2$ whence $j \geq 2$. This then gives us by lemma 5.3.2 that $(-\omega)^j(y-1) = y' \in I$, and by $IH(m - F_{\iota(m)})$ we have $P_{m-F_{\iota(m)}}((-\omega)^{\iota(m-F_{\iota(m)})}y') \geq 1$ with equality only for $m = F_{\iota(m)}$.

It follows that $P_m((-\omega)^{\iota(m)}y) \geq P_{F_{\iota(m)}}((-\omega)^{\iota(m)}y)$ with equality only for $m = F_{\iota(m)}$. But $\iota(m) \geq 2$ and so we can apply the Fibonacci Hypothesis to obtain $P_m((-\omega)^{\iota(m)}y) \geq P_{F_{\iota(m)}}((-\omega)^{\iota(m)}y) > 1$ which establishes the induction. □

5.3.3 Conditions for the Fibonacci Hypothesis

In the previous section we showed that $P_k((-\omega)^n y)$ is bounded below by $P_{F_n}((-\omega)^n y)$ (where n is the Fibonacci index of k), if the Fibonacci Hypothesis holds, ie if $P_{F_n}((-\omega)^n y) > 1$ for $y \in I = [-\omega, \omega^2]$ and $n \geq 2$. In this section we show that the hypothesis holds if a certain other constraint holds at the single point $y = 0$.

Definition 5.3.5. If a real function f is defined on a compact interval $[a, b]$ and also achieves its infimum on the interval at either a or b, we say f is *ENDPOINT MINIMAL* on $[a, b]$

Note that if f is concave on an interval, it is also endpoint minimal on that interval. However there are many more complex functions which also have the property.

Lemma 5.3.6. *Let $[a,b] \subseteq [-\omega^{-1}, 1]$ then on this interval, for $n \geq 0$, $0 \leq k \leq F_{n+1} - 1$, the function $P_k((-\omega)^n y)$ has a concave logarithm and is endpoint minimal.*

Proof. For $k = 0$ the result is trivial, and so is true if $n = 0, 1$. We now assume $k \geq 1$ and $n \geq 2$. Put $L(y) = \log P_k((-\omega)^n y)$, then $L''(y) = -(-\omega)^{2n}\pi^2 \sum_{r=1}^{k} \csc^2 \pi(r\omega + (-\omega)^n y) < 0$ but with singularities whenever $\{r\omega + (-\omega)^n y\} = 0$. Hence $L(y)$ is concave on any interval without singularities, and so is endpoint minimal on that interval. We now examine the singularities closest to $y = 0$, knowing that $L(y)$ is endpoint minimal on any interval between such singularities.

This means we want to find the smallest values of $|y|$ satisfying $\{k\omega + (-\omega)^n y\} = 0$ for some $1 \leq k \leq F_{n+1} - 1$.

For $k \geq 1, n \geq 3$, the smallest values of $\|k\omega\|$ in the range $1 \leq k \leq F_{n+1} - 1$ are $\left\{-(-\omega)^n\right\}$ and $\left\{-(-\omega)^{n-1}\right\}$ occurring at $k = F_n, F_{n-1}$ respectively. Hence the smallest magnitudes of y which give a singularity are given by $\{-(-\omega)^p + (-\omega)^n y\} = 0$ for $p = n, n-1$, and this gives $y = 1, -\omega^{-1}$ respectively. For $n = 2$ the only value of k in range is 1, giving a singularity at $y = 1$. Other singularities then occur when $\left\{-(-\omega)^2 + (-\omega)^2 y\right\} = 0 = \left\{\omega^2(y-1)\right\}$ giving $y = 1 + r\omega^{-2}$. The next closest singularity is given by $r = -1$, giving $y = 1 - 1/\omega^2 = -\omega^{-1}$.

Combining all the results, for $n \geq 0$ and $[a, b] \subseteq (-\omega^{-1}, 1)$, $L(y)$ is endpoint minimal on $[a, b]$. Taking exponents gives us that $P_k((-\omega)^n y)$ is also endpoint minimal on $[a, b]$. The result extends to include the interval endpoints (ie to $[a, b] \subseteq [-\omega^{-1}, 1]$) by the continuity of $P_k((-\omega)^n y)$. □

We use this result to reduce the Fibonacci Hypothesis to a constraint on the point $y = \omega^2$:

Lemma 5.3.7. *Suppose for $n \geq 2$ that $P_{F_n}((-\omega)^{n+2}) > 1$, then the Fibonacci Hypothesis holds, ie for $n \geq 2$ we have $P_{F_n}((-\omega)^n y) > 1$ for $y \in I$*

Proof. Since $I \subseteq [-\omega^{-1}, 1]$ the previous lemma gives us that $P_{F_n}((-\omega)^n y)$ is endpoint minimal on I and so $\inf P_{F_n}((-\omega)^n y)$ is achieved at an endpoint of $I = [-\omega, \omega^2]$. If the endpoint is ω^2, the infimum is $P_{F_n}((-\omega)^n \omega^2)$ and for $n \geq 2$ the result follows immediately by hypothesis. If the endpoint is $-\omega$ the infimum is $P_{F_n}((-\omega)^{n+1})$, and then:

$$\begin{aligned} P_{F_n}((-\omega)^{n+1}) &= P_{F_{n-1}}((-\omega)^{n+1}) P_{F_{n-2}}((-\omega)^{n+1} - (-\omega)^{n-1}) \\ &= P_{F_{n-1}}((-\omega)^{n-1}\omega^2) P_{F_{n-2}}((-\omega)^{n-2}\omega^2) \end{aligned}$$

For $n \geq 4$, the hypothesis gives us that both terms in the product are greater than 1, and hence also $P_{F_n}((-\omega)^{n+1}) > 1$. The result is therefore established apart from the cases when the endpoint is $-\omega$ and $n \in \{2, 3\}$. For $n = 2$, $P_{F_n}((-\omega)^{n+1}) = 2\sin\pi(\omega - \omega^3) > 1$. And for $n = 3$, $P_{F_n}((-\omega)^{n+1}) = \left|2\sin\pi(\omega + \omega^4) 2\sin\pi(2\omega + \omega^4)\right| > 1$, and all cases are covered. □

Finally we express the condition at $y = \omega^2$ as a condition at $y = 0$, ie a condition on $P_{F_n}(0)$

Lemma 5.3.8. *For $n \geq 1$, $P_{F_n-1}((-\omega)^n y)$ is symmetric about $y = 1/2$ and for $y \in [0, 1]$ we have $P_{F_n-1}(0) \leq P_{F_n-1}((-\omega)^n y)$. Further for $n \geq 2$, $P_{F_n}((-\omega)^{n+2}) > \omega(1 - (\pi^2/6)\omega^{2n+2}) P_{F_n}(0)$.*

Proof. For $n \geq 1$ and any y, we have:

$$P_{F_n-1}((-\omega)^n y) = \left|\prod_{r=1}^{F_n-1} 2\sin\pi(r\omega + (-\omega)^n y)\right| = \left|\prod_{r=1}^{F_n-1} 2\sin\pi((F_n - r)\omega + (-\omega)^n y)\right|$$
$$= \left|\prod_{r=1}^{F_n-1} 2\sin\pi(r\omega + (-\omega)^n - (-\omega)^n y)\right| = P_{F_n-1}((-\omega)^n(1-y))$$

Hence $P_{F_n-1}((-\omega)^n y)$ is symmetric about 1/2. But by lemma 5.3.6, it is also endpoint minimal on $[0,1]$. It follows that for $y \in [0,1]$,

$$P_{F_n-1}((-\omega)^n y) \geq P_{F_n-1}(0) = P_{F_n-1}((-\omega)^n) \tag{5.1}$$

Hence for $y \in [0,1]$, and using $\{F_n\omega\} = \{-(-\omega)^n\}$:

$$\begin{aligned} P_{F_n}((-\omega)^n y) &= P_{F_n-1}((-\omega)^n y)\,|2\sin\pi((-\omega)^n y + F_n\omega)| \\ &= P_{F_n-1}((-\omega)^n y)\,|2\sin\pi((-\omega)^n y - (-\omega)^n)| \\ &\geq P_{F_n-1}(0)\, 2\sin\pi\omega^n(1-y) \\ &= (P_{F_n}(0)/|2\sin\pi(0 + F_n\omega|)\ 2\sin\pi\omega^n(1-y) \\ &= (P_{F_n}(0)/\sin\pi\omega^n)\ \sin\pi\omega^n(1-y) \end{aligned}$$

Now using $x > \sin x > x - x^3/6$ for $0 < x < \pi/2$ gives for $y \in (0,1]$:

$$\begin{aligned} P_{F_n}((-\omega)^n y) &> (P_{F_n}(0)/(\pi\omega^n))\ (\pi\omega^n(1-y) - (\pi\omega^n(1-y))^3/6) \\ &= (1-y)\left(1 - (\pi\omega^n(1-y))^2/6\right) P_{F_n}(0) \end{aligned}$$

The final result follows on setting $y = \omega^2$. □

In particular this means that lemma 5.3.7 is satisfied, and hence the Fibonacci Hypothesis (5.3.3) holds, if we can show for $n \geq 2$ that:

$$P_{F_n}(0) > 1/\left(\omega(1 - (\pi^2/6)\omega^{2n+2})\right) \geq (1+\omega)/(1 - (\pi^2/6)\omega^6) = P \approx 1.781 \tag{5.2}$$

where we write P to denote the constant. We have therefore established the Hypothesis if we can show $P_{F_n}(0) > P$ for $n \geq 2$.

5.3.4 Bounds on P_{F_n}

At this point we have shown for $y \in I$ and $k \geq 1$ (which means $\iota(k) \geq 2$), that $P_k((-\omega)^{\iota(k)}y) \geq P_{F_{\iota(k)}}((-\omega)^{\iota(k)}y) > 1$ if for $n \geq 2$ we have $P_{F_n}(0) > P = (1+\omega)/(1 - (\pi^2/6)\omega^6) \approx 1.781$. In this

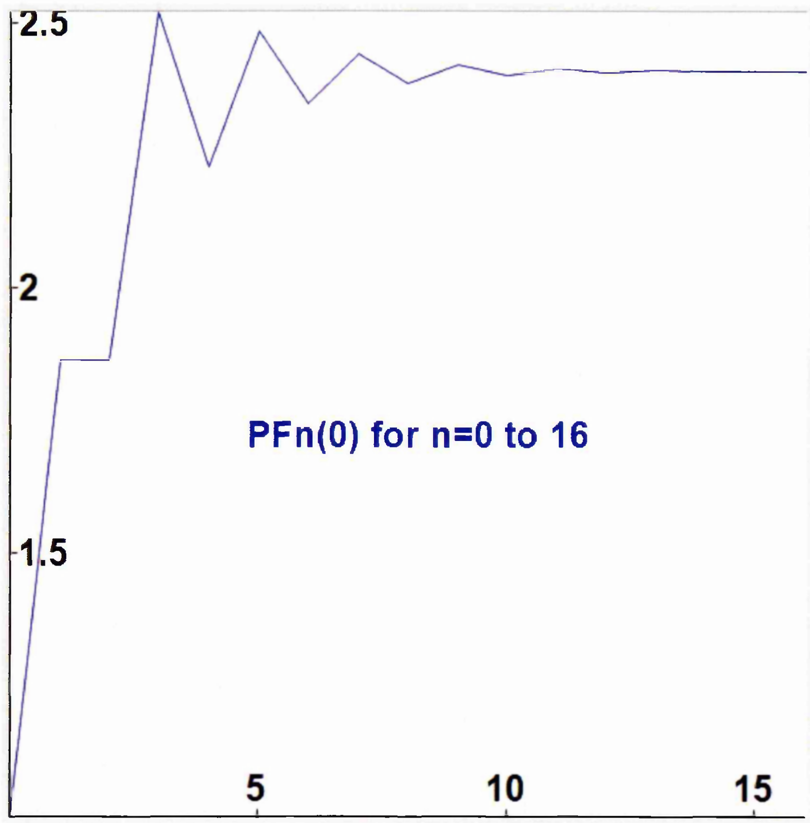

Figure 5.3.1: $P_{F_n}(0)$ for $n = 0 \ldots 16$ showing values that are "obviously" $\geq P_{F_1}(0)$ but a formal proof takes effort.

section we show that this condition holds. Although the result seems evident from inspection of a graph such as Fig 5.3.1 (given that $P_{F_1}(0) = P_{F_2}(0) \approx 1.864$), a formal proof seems remarkably arduous but we have not been able to find simpler.

Our strategy is to develop an increasing sequence of lower bounds for $P_{F_n} := P_{F_n}(0)$, but this proves difficult to do directly, despite the alternating character of the sequence P_{F_n} suggested by Figure 5.3.1. This is because direct estimates for P_{F_n} involve irregular oscillations deriving from the irregular nature of the sequence $\{rF_{n-1}/F_n\}$ (which in turn derives from the irregularity of the sequence of the residues of $rF_{n-1} \bmod F_n$). Fortunately it turns out that truncations of the sequences we developed in chapter 4 contain very similar irregularities, and by taking ratios (and after some tedious manipulation) we can effectively cancel the irregular behaviours and obtain monotone behaviour, and this proves to be the breakthrough step. There remains a slightly delicate question of recovering the necessary information about P_{F_n} from the ratios, but the monotonicity of both upper and lower bounds of the ratios allows this to be done (see 5.3.4.5) .

5.3.4.1 Preliminary results

We will exploit a number of results proved in chapter 4. In particular we will reuse the decomposition $P_{F_n} = A_n B_n C_n \to c$ where, for $n \geq 3$:

$$\begin{aligned}
A_n &= 2F_n \sin \pi\omega^n \to A_\infty = \frac{2\pi}{\sqrt{5}} \\
B_n &= \prod_{t=1}^{(F_n-1)/2} \left(\frac{s_{nt}}{2\sin\pi\frac{t}{F_n}}\right)^2 \to B_\infty = \prod_{t=1}^{\infty}\left(1-\frac{\xi_{\infty t}}{\sqrt{5}t}\right)^2 \\
C_n &= \prod_{t=1}^{(F_n-1)/2}\left(1-\frac{s_{n0}^2}{s_{nt}^2}\right) \to C_\infty = \prod_{t=1}^{\infty}\left(1-\frac{1}{20t^2\left(1-\frac{\xi_{\infty t}}{\sqrt{5}t}\right)^2}\right)
\end{aligned}$$

and where $s_{nt} = 2\sin\pi\,(t/F_n - \omega^n\xi_{nt})$, $\xi_{nt} = \left\{\frac{F_{n-1}}{F_n}t\right\} - \frac{1}{2}$, and $\xi_{\infty t} = \{t\omega\} - \frac{1}{2}$. See also (4.3) for the definition of the notation $\Pi_1^{m/2}$.

We will call a pair of real numbers $x \leq y$ a BRACKET for z if $x \leq z \leq y$. For our work in this section we need some simple brackets for the products B_n, C_n, which we now derive. Note that if F_n is odd, then $\lfloor (F_n-1)/2 \rfloor = \lfloor F_n/2 \rfloor$, whilst if F_n is even then $\lfloor (F_n-1)/2 \rfloor < \lfloor F_n/2 \rfloor$. We now develop brackets for B_n, C_n.

For F_n even and $t = F_n/2$, we have $\frac{s_{nt}}{2\sin\pi\frac{t}{F_n}} = \frac{2}{2} = 1$, and so the existence of the term $t = F_n/2$ makes no difference to the product B_n. This gives us the bracket:

$$\prod_{t=1}^{\lfloor (F_n-1)/2 \rfloor}\left(\frac{s_{nt}}{2\sin\pi\frac{t}{F_n}}\right)^2 = B_n = \prod_{t=1}^{\lfloor F_n/2 \rfloor}\left(\frac{s_{nt}}{2\sin\pi\frac{t}{F_n}}\right)^2 \tag{5.3}$$

And for $1 \leq t \leq F_n - 1$ we have $s_{nt} > s_{n0}$, and so $0 < 1 - \frac{s_{n0}^2}{s_{nt}^2} < 1$. This gives us the bracket (with equality for F_n odd, and inequality for F_n even):

$$\prod_{t=1}^{\lfloor (F_n-1)/2 \rfloor}\left(1-\frac{s_{n0}^2}{s_{nt}^2}\right) \geq C_n \geq \prod_{t=1}^{\lfloor F_n/2 \rfloor}\left(1-\frac{s_{n0}^2}{s_{nt}^2}\right) \tag{5.4}$$

Now $P_{F_2} = 2\sin\pi\omega \approx 1.864 < c$ and so $\inf_{n\geq 2} P_{F_n} \leq P_{F_2} < c$. Since $P_{F_n} \to c$ this infimum must be achieved at some $2 \leq n_0 < \infty$. In fact we will show $n_0 = 2$.

To find the infimum we will analyse each of the constituent terms of $P_{F_n}/c = (A_n/A_\infty)(B_n/B_\infty)(C_n/C_\infty)$ in turn. We will use the following results:

From (4.6), for $n \geq 0$

$$F_n\omega^n = \frac{1}{\sqrt{5}}(1-(-1)^n\omega^{2n}) \tag{5.5}$$

From lemma 4.3.3 part (5), we have, for $1 \leq t \leq F_n - 1$:

$$\xi_{nt} = \xi_{\infty t} + t(-\omega)^n/F_n \tag{5.6}$$

Finally we also need a simple lemma derived from previous results:

Lemma 5.3.9. *For any integers* $1 < p < q \leq \infty$ *we have*

$$\left| \sum_{t=p+1}^{q} \frac{\xi_{\omega t}}{t} \right| < \frac{3}{2\log(2+\omega)} \left(\frac{2+\log p}{p} - \frac{1}{q} \right)$$

Proof. We start by using partial summation to obtain:

$$\sum_{t=p+1}^{q} \frac{\xi_{\omega t}}{t} = \sum_{t=p+1}^{q-1} \left((\frac{1}{t} - \frac{1}{t+1}) \sum_{s=p+1}^{t} \xi_{\omega s} \right) + \frac{1}{q} \sum_{s=p+1}^{q} \xi_{\omega s} \tag{5.7}$$

Now:

$$\sum_{s=p+1}^{t} \xi_{\omega s} = \sum_{s=p+1}^{t} \{s\omega\} - \frac{1}{2} = \sum_{s'=0}^{t-p-1} \{(p+1)\omega + s'\omega\} - \frac{1}{2}$$

But now from theorem 2.5.4 we have $\left|\sum_{s'=0}^{t-p-1} \{(p+1)\omega + s'\omega\} - \frac{1}{2}\right| \leq \frac{3}{2} O(t-p)$ where $O_\alpha(n)$ is the Ostrowski digit sum of $n \geq 0$ (see 2.5.2). And from (4.59) we have, for $t - p \geq 1$, $O(t-p) \leq \lfloor (\log(t-p)+1)/\log(2+\omega) \rfloor$. Hence for $t \geq p+1$:

$$\begin{aligned} \left| \sum_{s=p+1}^{t} \xi_{\omega s} \right| &\leq \frac{3}{2} \left\lfloor \frac{(\log(t-p)+1)}{\log(2+\omega)} \right\rfloor \\ &< \frac{3(\log(t-p)+1)}{2\log(2+\omega)} \end{aligned} \tag{5.8}$$

We now use this result in (5.7) to obtain (using the premise $p > 1$):

$$\begin{aligned} \left| \sum_{t=p+1}^{q} \frac{\xi_{\omega t}}{t} \right| &< \frac{3}{2\log(2+\omega)} \left(\sum_{t=p+1}^{q-1} \frac{(\log(t-p)+1)}{t(t+1)} + \frac{\log(q-p)+1}{q} \right) \\ &< \frac{3}{2\log(2+\omega)} \left(\int_{t=p}^{q-2} \frac{(\log t + 1)}{t^2} dt + \frac{1+\log q}{q} \right) \\ &= \frac{3}{2\log(2+\omega)} \left(\frac{\log p}{p} - \frac{\log(q-2)}{q-2} + \frac{2}{p} - \frac{2}{q-2} + \frac{1+\log q}{q} \right) \\ &< \frac{3}{2\log(2+\omega)} \left(\frac{2+\log p}{p} - \frac{1}{q} \right) \end{aligned}$$

which completes the proof. □

Product Inequalities We will need some product inequalities. First from lemma 4.3.2, if $|x_t| < 1$ for $t = 1 \ldots n$, and also $\sum_{t=1}^{n} |x_t| < 1$:

$$1 - \sum_{t=1}^{n} |x_t| \leq \prod_{t=1}^{n} (1 + x_t) \leq \frac{1}{1 - \sum_{t=1}^{n} |x_t|} \tag{5.9}$$

Secondly, from lemma 4.7.5, if $1 \geq x_t > -0.683$ for $t = 1 \ldots n$:

$$\exp\left(\sum_{t=1}^{n} x_t - \sum_{t=1}^{n} x_t^2\right) \leq \prod_{t=1}^{n}(1 + x_t) \leq \exp\sum_{t=1}^{n} x_t \tag{5.10}$$

Finally writing $S^1 = \left|\sum_{t=1}^n x_t\right|$ and $S^2 = \sum_{t=1}^n x_t^2$ we can relax (5.10) to obtain:

$$\exp\left(-\left(S^1 + S^2\right)\right) \leq \prod_{t=1}^{n}(1 + x_t) \leq \exp\, S^1 \tag{5.11}$$

In our cases, (5.10) generally produces the best results. However what is far more important from our point of view is the ease with which the various sums can be calculated. In particular in many of our cases the sign of $\sum_{t=1}^n x_t$ is not easily determined, which forces us away from (5.10) to (5.11). However again $\sum_{t=1}^n |x_t|$ is typically much easier to calculate than S^1, S^2, and we will therefore use (5.9) when this produces results adequate for our purposes.

We now develop brackets for each of the terms in the product $(A_n/A_\infty)\,(B_n/B_\infty)\,(C_n/C_\infty)$

5.3.4.2 A bracket for A_n/A_∞

This term is the simplest term to analyse. From (5.5):

$$A_n/A_\infty = (2F_n \sin \pi\omega^n)\,/\,\left(2\pi/\sqrt{5}\right) = (\omega^{-n}/\pi)\left(1 - (-1)^n\omega^{2n}\right)\sin \pi\omega^n$$

and then using $x > \sin x > x - x^3/6$ for $x > 0$, we get, for $n \geq 0$:

$$\frac{A_n}{A_\infty} < \frac{\omega^{-n}}{\pi}\left(1 - (-1)^n\omega^{2n}\right)\pi\omega^n < 1 + \omega^{2n} \tag{5.12}$$

$$\frac{A_n}{A_\infty} > \frac{\omega^{-n}}{\pi}\left(1 - (-1)^n\omega^{2n}\right)\pi\omega^n(1 - \frac{\pi^2\omega^{2n}}{6}) > 1 - \omega^{2n}(1 + \frac{\pi^2}{6}) \tag{5.13}$$

Note that the bracket converges monotonically to 1 with n.

5.3.4.3 A bracket for B_n/B_∞

Recall that

$$B_n/B_\infty = \prod_{t=1}^{(F_n-1)/2}\left(\frac{s_{nt}}{2\sin \pi \frac{t}{F_n}}\right)^2 \Bigg/ \prod_{t=1}^{\infty}\left(1 - \frac{\xi_{\infty t}}{\sqrt{5}t}\right)^2$$

We will need a more useful expression for the terms $s_{nt}/(2\sin \pi t/F_n)$ occurring in B_n. Equation (5.6) gives $\xi_{nt} - \xi_{\infty t} = t(-\omega)^n/F_n$ for $1 \leq t \leq F_n - 1$, so that:

$$\begin{aligned}
\frac{s_{nt}}{2\sin \pi \frac{t}{F_n}} &= \frac{2\sin \pi\,(t/F_n - \omega^n\xi_{nt})}{2\sin \pi \frac{t}{F_n}} = \cos \pi\omega^n\xi_{nt} - \cot \frac{\pi t}{F_n} \sin \pi\omega^n\xi_{nt} \\
&= 1 - \left(2\sin^2\left(\frac{1}{2}\pi\omega^n\xi_{nt}\right) + \cot \frac{\pi t}{F_n} \sin\left(\pi\omega^n\xi_{nt}\right)\right)
\end{aligned}$$

which we write as $1 - b_{nt}$. We also write $b_{\infty t} = \frac{\xi_{\infty t}}{\sqrt{5}t}$, and $M = \lfloor F_n/2 \rfloor$ then using (5.3) we can write:

$$B_n/B_\infty = \prod_{t=1}^{M}(1 - b_{nt})^2 \Big/ \prod_{t=1}^{\infty}(1 - b_{\infty t})^2$$

Obtaining a suitable estimate for this product proves quite a delicate operation. We will need the precision of the exponential form of the Product Inequality 5.11 to obtain a bracket, and using this requires some arduous manipulation. Even then, a straightforward application of the Product Inequality proves ineffective: it leads to a non-convergent bracket, and this does not meet our needs. Instead we divide the product into 3 sub-products, and use differing techniques appropriate to each one.

Overview of the estimating process We put $N = \left\lfloor \omega^{-n/2} \right\rfloor$ and require $1 \leq N \leq M$, which is satisfied for $n \geq 4$. We can then write:

$$B_n/B_\infty = \left(\prod_{t=1}^{N} 1 - \frac{b_{nt} - b_{\infty t}}{1 - b_{\infty t}}\right)^2 \left(\frac{\prod_{t=N+1}^{M}(1 - b_{nt})}{\prod_{t=N+1}^{\infty}(1 - b_{\infty t})}\right)^2 = \left(P_\alpha \frac{P_\beta}{P_\gamma}\right)^2 \tag{5.14}$$

and we now analyse each of the products $P_\alpha, P_\beta, P_\gamma$.

In order to use the exponential form of the Product Inequality (5.11) with a product $\prod(1 + x_t)$, we need to estimate the sums $S_x^1 = |\sum x_i|$ and $S_x^2 = \sum x_i^2$. The inequality then gives us:

$$\exp\left(-(S_x^1 + S_x^2)\right) < \prod(1 + x_t) < \exp S_x^1$$

We proceed to estimate the sums S_x^1, S_x^2 for each of $P_\alpha, P_\beta, P_\gamma$.

Bounds for P_α First note that since $|\xi_{nt}| < 1/2$ we have $1 - b_{\infty t} = 1 - \frac{\xi_{nt}}{\sqrt{5}t} > 1 - \frac{1}{2\sqrt{5}}$. To apply (5.11) we will need to estimate:

$$S_\alpha^1 = \left|\sum_{t=1}^{N} \frac{b_{nt} - b_{\infty t}}{1 - b_{\infty t}}\right| < \left(\frac{1}{1 - \frac{1}{2\sqrt{5}}}\right) \sum_{t=1}^{N} |b_{nt} - b_{\infty t}| \tag{5.15}$$

$$S_\alpha^2 = \sum_{t=1}^{N} \left(\frac{b_{nt} - b_{\infty t}}{1 - b_{\infty t}}\right)^2 < \left(\frac{1}{1 - \frac{1}{2\sqrt{5}}}\right)^2 \sum_{t=1}^{N} |b_{nt} - b_{\infty t}|^2 \tag{5.16}$$

It turns out that obtaining a useful estimate for S_α^1 depends upon both exploiting the sign changes of $\xi_{nt}, \xi_{\infty t}$ and the fact that these two values always have the same sign. To enable us to preserve the necessary information we develop an auxiliary representation of the cot and sin functions as follows:

For $0 \leq x \leq 1$ we have $x(1 - x^2/6) \leq \sin x \leq x$ so that we can write $\sin x = x(1 - \sigma x^2)$ for some $0 < \sigma < 1/6$. Note that σ depends on x, but this is immaterial for our argument. Also for $-1 \leq x < 0$ we have $x(1 - x^2/6) > \sin x > x$ so that we can write $\sin x = x(1 - \sigma x^2)$ where again $0 < \sigma < 1/6$. Hence for $|x| \leq 1$ we have $\sin x = x(1 - \sigma x^2)$ for some $0 < \sigma < 1/6$ which depends on x. Similarly $(1 - x^2/2)/x < \cot x < 1/x$ for $0 < x \leq 1$ and by the same reasoning we obtain $\cot x = (1 - \kappa x^2)/x$ for $0 \neq |x| \leq 1$ for some $0 < \kappa < 1/2$ which depends on x.

We can now begin to analyse (5.15). First we note:

$$b_{nt} - b_{\infty t} = 2\sin^2 \frac{1}{2}\pi\omega^n \xi_{nt} + \cot\frac{\pi t}{F_n}\sin \pi\omega^n\xi_{nt} - \frac{\xi_{\infty t}}{\sqrt{5}t} \tag{5.17}$$

We now investigate the size of the cotangent term in (5.17). We use equations (5.6) $\xi_{nt} = \xi_{\infty t} + t(-\omega)^n/F_n$ and (5.5) $F_n\omega^n = \frac{1}{\sqrt{5}}(1 - (-1)^n\omega^{2n})$ to obtain for $0 < \pi t/F_n \leq 1$ (giving $1 \leq t \leq F_n/\pi$ which requires $n \geq 5$):

$$\begin{aligned}
\cot\frac{\pi t}{F_n}\sin\pi\omega^n\xi_{nt} &= \frac{F_n}{\pi t}\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\pi\omega^n\xi_{nt}\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right) \\
&= \frac{(1-(-1)^n\omega^{2n})}{\sqrt{5}t}\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(\xi_{\infty t} + \frac{t(-\omega)^n}{F_n}\right)\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right)
\end{aligned} \tag{5.18}$$

Hence:

$$\begin{aligned}
\cot\frac{\pi t}{F_n}\sin\pi\omega^n\xi_{nt} - \frac{\xi_{\infty t}}{\sqrt{5}t} \quad &= \quad \frac{(1-(-1)^n\omega^{2n})}{\sqrt{5}t}\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right)\left(\xi_{\infty t} + \frac{t(-\omega)^n}{F_n}\right) \\
&\quad - \frac{\xi_{\infty t}}{\sqrt{5}t} \\
&= \quad \frac{(1-(-1)^n\omega^{2n})}{\sqrt{5}t}\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right)\frac{t(-\omega)^n}{F_n} \\
&\quad + \frac{(1-(-1)^n\omega^{2n})}{\sqrt{5}t}\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right)\xi_{\infty t} - \frac{\xi_{\infty t}}{\sqrt{5}t} \\
&= \quad \frac{(-\omega)^n}{F_n}\frac{(1-(-1)^n\omega^{2n})}{\sqrt{5}}\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right) \\
&\quad + \frac{\xi_{\infty t}}{\sqrt{5}t}\left((1-(-1)^n\omega^{2n})\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right) - 1\right)
\end{aligned} \tag{5.19}$$

We now examine the coefficient of $\frac{\xi_{\infty t}}{\sqrt{5}t}$ in (5.19) which we designate Z with component terms U, V, W giving us:

$$Z = (1-(-1)^n\omega^{2n})\left(1 - \kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2\right) - 1 = UVW - 1$$

Now for $n \geq 5$, $1 \leq t \leq F_n/\pi$ we have $0 < V, W < 1$ and $U = 1 \pm \omega^{2n} > 0$. Hence the lower bound of Z is achieved in the case $U = 1 - \omega^{2n}$, and we get (using $0 < VW < 1$):

$$Z > -\kappa\left(\frac{\pi t}{F_n}\right)^2 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2 - \omega^{2n}\left(1-\kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1-\sigma\left(\pi\omega^n\xi_{nt}\right)^2\right) \tag{5.20}$$

$$> -\kappa\left(\frac{\pi t}{F_n}\right)^2 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2 - \omega^{2n}$$

The upper bound is achieved in the case $U = 1 + \omega^{2n}$ and then (again using $0 < VW < 1$):

$$Z < (1+\omega^{2n})\left(1-\kappa\left(\frac{\pi t}{F_n}\right)^2\right)\left(1-\sigma\left(\pi\omega^n\xi_{nt}\right)^2\right) - 1 \tag{5.21}$$

$$< -\kappa\left(\frac{\pi t}{F_n}\right)^2 - \sigma\left(\pi\omega^n\xi_{nt}\right)^2 + \kappa\left(\frac{\pi t}{F_n}\right)^2\sigma\left(\pi\omega^n\xi_{nt}\right)^2 + \omega^{2n}$$

$$< -\kappa\left(\frac{\pi t}{F_n}\right)^2\left(1-\sigma\left(\pi\omega^n\xi_{nt}\right)^2\right) - \sigma\left(\pi\omega^n\xi_{nt}\right)^2 + \omega^{2n}$$

Comparing this with (5.20) gives (using $0 < \sigma\left(\pi\omega^n\xi_{nt}\right)^2 < 1$ and $\kappa < 1/2$, $\sigma < 1/6$):

$$|Z| < \kappa\left(\frac{\pi t}{F_n}\right)^2 + \sigma\left(\pi\omega^n\xi_{nt}\right)^2 + \omega^{2n}$$

$$< \kappa\left(\frac{\pi t}{F_n}\right)^2 + \omega^{2n}\left(1+\sigma\pi^2\omega^{2n}\frac{1}{4}\right)$$

$$< \frac{1}{2}\left(\frac{\pi t}{F_n}\right)^2 + \omega^{2n}\left(1+\frac{\pi^2\omega^{2n}}{24}\right) \tag{5.22}$$

We now estimate the first term in (5.15), using $\sin^2 x \le x^2$, $|\xi_{nt}| < 1/2$ and $|b_{\infty t}| < 1/(2\sqrt{5}t)$ to give $2\sin^2\frac{1}{2}\pi\omega^n\xi_{nt} < \pi^2\omega^{2n}/8$. We can now combine this result with (5.22) and (5.19) back in (5.17) to obtain (using $\kappa, \sigma > 0$, $|\xi_{\infty t}| < 1/2$, $F_n < \omega^{-n}(1-\omega^{2n})/\sqrt{5}$, $t \ge 1$,):

$$|b_{nt} - b_{\infty t}| = \left|2\sin^2\frac{1}{2}\pi\omega^n\xi_{nt} + \cot\frac{\pi t}{F_n}\sin\pi\omega^n\xi_{nt} - \frac{\xi_{\infty t}}{\sqrt{5}t}\right| \tag{5.23}$$

$$< \frac{\pi^2\omega^{2n}}{8} + \frac{\omega^n\left(1+\omega^{2n}\right)}{F_n\sqrt{5}} + \frac{1}{2\sqrt{5}t}\left(\frac{1}{2}\left(\frac{\pi t}{F_n}\right)^2 + \omega^{2n}\left(1+\frac{\pi^2\omega^{2n}}{24}\right)\right)$$

$$\le \omega^{2n}\left(\frac{\pi^2}{8} + \frac{1+\omega^{2n}}{1-\omega^{2n}} + \frac{1}{2\sqrt{5}}\left(1+\frac{\pi^2\omega^{2n}}{24}\right)\right) + \frac{\sqrt{5}\pi^2\omega^{2n}t}{4(1-\omega^{2n})^2}$$

$$= \omega^{2n}\left(K_1(n) + K_2(n)t\right)$$

where K_1, K_2 are both $O(1)$. We can now evaluate (5.15) by summing the expression above for $1 \le t \le N$ to obtain:

$$S_\alpha^1 < \left(\frac{1}{1-\frac{1}{2\sqrt{5}}}\right)\omega^{2n}\left(K_1N + \frac{1}{2}K_2N(N+1)\right) \tag{5.24}$$

$$= O(\omega^n)$$

We now turn to consider S_α^2. We can use (5.23) in (5.16) to obtain:

$$\begin{aligned} S_\alpha^2 &< \left(\frac{1}{1-\frac{1}{2\sqrt{5}}}\right)^2 \sum_{t=1}^{N} \omega^{4n}(K_1+K_2t)^2 \\ &= \left(\frac{1}{1-\frac{1}{2\sqrt{5}}}\right)^2 \omega^{4n}\left(K_1^2N + K_1K_2N(N+1) + \frac{1}{6}K_2^2N(N+1)(2N+1)\right) \\ &= O(\omega^{5n/2}) \end{aligned}$$

We can finally apply the Product Inequality to estimate P_α as:

$$\exp\left(-(S_\alpha^1 + S_\alpha^2)\right) < P_\alpha < \exp S_\alpha^1 \tag{5.25}$$

where $S_\alpha^1 = O(\omega^n)$ and $S_\alpha^2 = O(\omega^{5n/2})$.

Bounds for P_γ We now turn to the product P_γ(as it is simpler than P_β). Now

$$P_\gamma = \prod_{t=N+1}^{\infty}(1-b_{\infty t}) = \prod_{t=N+1}^{\infty}\left(1-\frac{\xi_{\infty t}}{\sqrt{5}t}\right) \tag{5.26}$$

and we proceed to estimate S_γ^1 which is easily done using lemma 5.3.9:

$$S_\gamma^1 = \left|\sum_{t=N+1}^{\infty}\frac{\xi_{\infty t}}{\sqrt{5}t}\right| < \frac{3}{2\sqrt{5}\log(2+\omega)}\frac{2+\log N}{N} = O(n\omega^{n/2}) \tag{5.27}$$

And also

$$S_\gamma^2 = \left(\sum_{t=N+1}^{\infty}\frac{\xi_{\infty t}}{\sqrt{5}t}\right)^2 < \frac{1}{20}\sum_{t=N+1}^{\infty}\frac{1}{t^2} < \frac{1}{20}\frac{1}{N} = O(\omega^{n/2}) \tag{5.28}$$

We can then apply (5.11) to estimate $P_\gamma$in (5.14) as:

$$\exp\left(-(S_\gamma^1 + S_\gamma^2)\right) < P_\gamma < \exp S_\gamma^1$$

where $S_\gamma^1 = O(n\omega^{n/2})$ and $S_\gamma^2 = O(\omega^{n/2})$.

Bounds for P_β Finally we turn to estimating:

$$P_\beta = \prod_{t=N+1}^{M}(1-b_{nt})$$

In this case:

$$\begin{aligned}
S^1_\beta &= \left| \sum_{t=N+1}^{M} 2\sin^2 \frac{1}{2}\pi\omega^n \xi_{nt} + \cot\frac{\pi t}{F_n} \sin \pi\omega^n \xi_{nt} \right| \\
&< \sum_{t=N+1}^{M} 2\sin^2 \frac{1}{2}\pi\omega^n \xi_{nt} + \left| \sum_{t=N+1}^{M} \cot\frac{\pi t}{F_n} \sin \pi\omega^n \xi_{nt} \right| \\
&= S^1_{\beta 1} + S^1_{\beta 2} \qquad (5.29)
\end{aligned}$$

The term $S^1_{\beta 1}$ is again easy to estimate: $S^1_{\beta 1} < (M-N)\pi^2\omega^{2n}/8 = O(\omega^n)$.

We now turn to the sum $S^1_{\beta 2}$ in (5.29). In fact we have already proved in chapter 4 that this sum converges. We can follow the structure of this previous proof although we now have rather more work as we will need to track the size of the error terms.

We start with summation by parts:

$$\begin{aligned}
\sum_{t=N+1}^{M} \cot\frac{\pi t}{F_n} \sin\pi\omega^n\xi_{nt} &= \sum_{t=N+1}^{M-1} \left(\cot\left(\frac{\pi t}{F_n}\right) - \cot\left(\frac{\pi(t+1)}{F_n}\right)\right) \sum_{s=N+1}^{t} \sin\pi\left(\omega^n \xi_{ns}\right) \\
&\quad + \cot\left(\frac{\pi M}{F_n}\right) \sum_{s=N+1}^{M} \sin\pi\left(\omega^n\xi_{ns}\right) \qquad (5.30)
\end{aligned}$$

We first estimate the trailing term. Recall $M = \lfloor F_n/2 \rfloor$ so if F_n is even the trailing term is 0. Otherwise $M = (F_n - 1)/2$ and

$$\begin{aligned}
\cot\left(\frac{\pi M}{F_n}\right) \left| \sum_{s=N+1}^{M} \sin\pi\left(\omega^n\xi_{ns}\right) \right| &< \tan\frac{\pi}{2F_n} \left(\sum_{s=N+1}^{M} \frac{\pi\omega^n}{2} \right) \\
&< \frac{\pi/2F_n}{1 - \frac{1}{2}(\pi/2F_n)^2} \left(\frac{F_n}{2} \times \frac{\pi\omega^n}{2} \right) \\
&< \frac{\pi^2\omega^n}{8} \Bigg/ \left(1 - \frac{1}{2}\left(\frac{\pi}{2F_n}\right)^2 \right) \qquad (5.31)
\end{aligned}$$

We now estimate the first sine sum in (5.30), using $\sin x = x(1-\sigma x^2)$ for $|x| < 1$ and $0 < \sigma < 1/6$, and $|\xi_{ns}| < 1/2$:

$$\begin{aligned}
\left| \sum_{s=N+1}^{t} \sin\pi\left(\omega^n\xi_{ns}\right) \right| &= \left| \sum_{s=N+1}^{t} \pi\omega^n\xi_{ns} \left(1 - \sigma\left(\pi\omega^n\xi_{ns}\right)^2\right) \right| \qquad (5.32) \\
&< \pi\omega^n \left(\left| \sum_{s=N+1}^{t} \xi_{ns} \right| + \frac{\pi^2\omega^{2n}}{24}(t-N) \right)
\end{aligned}$$

Recall we write $O_\alpha(n)$ for the Ostrowski digit sum of n with respect to ω, which by lemma 4.6.2 is bounded for $n \geq 1$ by $(\log n + 1)/\log(2+\omega)$. But now using theorem 2.5.4 (and $1 \leq N < t \leq M$):

$$\begin{aligned}
\left|\sum_{s=N+1}^{t} \xi_{ns}\right| &= \left|\sum_{s=N+1}^{t}\left(\xi_{\infty s} + \frac{t(-\omega)^n}{F_n}\right)\right| \\
&\leq \left|\sum_{s'=1}^{t-N}\left(\{(s'+N)\omega\} - \frac{1}{2}\right)\right| + \sum_{s=N+1}^{t} \frac{t\omega^n}{F_n} \\
&< \frac{3}{2}O(t-N) + \frac{1}{2}t(t+1)\frac{\omega^n}{F_n} \\
&< \frac{3(\log(t-N)+1)}{2\log(2+\omega)} + \frac{M(M+1)\omega^n}{2F_n}
\end{aligned}$$

We now use these results in (5.32) to obtain:

$$\left|\sum_{s=N+1}^{t} \sin\pi\left(\omega^n \xi_{ns}\right)\right| < \pi\omega^n\left(\frac{3(\log(t-N)+1)}{2\log(2+\omega)} + \frac{M(M+1)\omega^n}{2F_n} + \frac{\pi^2\omega^{2n}M}{24}\right) \tag{5.33}$$

Finally we use (4.51) together with (5.31) and (5.33) in (5.30) to get our desired bound (also using $M \leq F_n/2$):

$$\begin{aligned}
S^1_{\beta 2} &< \sum_{t=N+1}^{M} \frac{\pi F_n}{4t^2}\pi\omega^n\left(\frac{3(\log(t-N)+1)}{2\log(2+\omega)} + \frac{M(M+1)\omega^n}{2F_n} + \frac{\pi^2\omega^{2n}M}{24}\right) + \frac{\pi^2\omega^n}{8}/\left(1-\frac{1}{2}\left(\frac{\pi}{2F_n}\right)^2\right) \\
&< \sum_{t=N+1}^{M} \frac{\pi^2(1+\omega^{2n})}{4\sqrt{5}t^2}\left(\frac{3(\log(t-1)+1)}{2\log(2+\omega)} + \frac{(M+1)\omega^n}{4} + \frac{\pi^2\omega^{2n}M}{24}\right) + \frac{\pi^2\omega^n}{8}/\left(1-\frac{1}{2}\left(\frac{\pi}{2F_n}\right)^2\right) \\
&< \frac{\pi^2(1+\omega^{2n})}{4\sqrt{5}}\int_N^{M-1}\frac{1}{t^2}\left(\frac{3(\log t+1)}{2\log(2+\omega)} + \frac{(M+1)\omega^n}{4} + \frac{\pi^2\omega^{2n}M}{24}\right)dt + \frac{\pi^2\omega^n}{8}/\left(1-\frac{1}{2}\left(\frac{\pi}{2F_n}\right)^2\right) \\
&< \frac{\pi^2(1+\omega^{2n})}{4\sqrt{5}}\left(\frac{3(2+\log N)}{2\log(2+\omega)N} + \frac{(M+1)\omega^n}{4N} + \frac{\pi^2\omega^{2n}M}{24N}\right) + \frac{\pi^2\omega^n}{8}/\left(1-\frac{1}{2}\left(\frac{\pi}{2F_n}\right)^2\right) \\
&= O\left(\frac{\log N}{N}\right)
\end{aligned} \tag{5.34}$$

Finally we estimate S^2_β:

$$\begin{aligned}
S^2_\beta &= \sum_{t=N+1}^{M}\left(2\sin^2\frac{1}{2}\pi\omega^n\xi_{nt} + \cot\frac{\pi t}{F_n}\sin\pi\omega^n\xi_{nt}\right)^2 \\
&\leq \sum_{t=N+1}^{M}\left(x_t^2 + 2\left|x_t y_t\right| + y_t^2\right)
\end{aligned}$$

and we now estimate each term:

$$
\begin{aligned}
\sum_{t=N+1}^{M} x_t^2 &< (M-N)\frac{\pi^4\omega^{4n}}{64} = O(\omega^{3n}) \\
\sum_{t=N+1}^{M} 2\,|x_t y_t| &< \frac{\pi^2\omega^{2n}}{4}\sum_{t=N+1}^{M}|y_t| < \frac{\pi^2\omega^{2n}}{4}\sum_{t=N+1}^{M}\frac{F_n}{\pi t}\frac{\pi\omega^n}{2} < \frac{\pi^2\omega^{2n}(1+\omega^{2n})}{8\sqrt{5}}\log\frac{M}{N} = O(n\omega^{2n}) \\
\sum_{t=N+1}^{M} y_t^2 &< \sum_{t=N+1}^{M}\left(\frac{F_n}{\pi t}\frac{\pi\omega^n}{2}\right)^2 < \frac{\left(1+\omega^{2n}\right)^2}{20}\sum_{t=N+1}^{M}\frac{1}{t^2} < \frac{\left(1+\omega^{2n}\right)^2}{20}\frac{1}{N} = O(\frac{1}{N})
\end{aligned}
$$

We can then apply (5.11) to estimate P_β in (5.14) as:

$$\exp\left(-(S_\beta^1 + S_\beta^2)\right) < P_\beta < \exp\, S_\beta^1$$

where $S_\beta^1 = O(n\omega^{n/2})$ and $S_\beta^2 = O(\omega^{n/2})$.

Final results We have now upper bounds converging monotonically to 0 for all the sums required to apply the Product Inequality for B_n/B_∞. We now collect the results together to obtain (noting that we require $n \geq 5$ for all parts to hold):

$$\exp\left(-2\left(\left(S_\alpha^1 + S_\alpha^2\right) + \left(S_\beta^1 + S_\beta^2\right) + S_\gamma^1\right)\right) < \frac{B_n}{B_\infty} = \left(P_\alpha\frac{P_\beta}{P_\gamma}\right)^2 < \exp\left(2\left(S_\alpha^1 + S_\beta^1 + \left(S_\gamma^1 + S_\gamma^2\right)\right)\right)$$

where:

$$S_\alpha^1 < \left(\frac{\omega^{2n}}{1-\frac{1}{2\sqrt{5}}}\right)\left(K_1 N + \frac{1}{2}K_2 N(N+1)\right)$$

$$S_\alpha^2 < \left(\frac{\omega^{2n}}{1-\frac{1}{2\sqrt{5}}}\right)^2 \times \left(K_1^2 N + K_1K_2N(N+1) + \frac{1}{6}K_2^2N(N+1)(2N+1)\right)$$

$$K_1 = \left(\frac{\pi^2}{8} + \frac{1+\omega^{2n}}{1-\omega^{2n}} + \frac{1}{2\sqrt{5}}\left(1 + \frac{\pi^2\omega^{2n}}{24}\right)\right)$$

$$K_2 = \frac{\sqrt{5}\pi^2}{4(1-\omega^{2n})^2}$$

$$S_\beta^1 < (M-N)\frac{\pi^2\omega^{2n}}{8} + \frac{\pi^2(1+\omega^{2n})}{4\sqrt{5}}\left(\frac{3(2+\log N)}{2\log(2+\omega)N} + \frac{(M+1)\omega^n}{4N} + \frac{\pi^2\omega^{2n}M}{24N}\right) + \frac{\pi^2\omega^n}{8}\Bigg/\left(1 - \frac{1}{2}\left(\frac{\pi}{2F_n}\right)^2\right)$$

$$S_\beta^2 < (M-N)\frac{\pi^4\omega^{4n}}{64} + \frac{\pi^2\omega^{2n}(1+\omega^{2n})}{8\sqrt{5}}\log\frac{M}{N} + \frac{\left(1+\omega^{2n}\right)^2}{20}\frac{1}{N}$$

$$S_\gamma^1 < \frac{3}{2\sqrt{5}\log(2+\omega)}\frac{2+\log N}{N}$$

$$S_\gamma^2 < \frac{1}{20N}$$

By inspection, all of these terms decrease monotonically to 0 with the slowest term having order $O(n\omega^{n/2}) = O(\log N/N)$ and so this bracket for B_n/B_∞ converges to 1 at least as fast as $N^{k/N}$ for some constant k.

5.3.4.4 A bracket for C_n/C_∞

We now turn to C_n/C_∞. Recall that we put $M = \lfloor F_n/2 \rfloor$. In this section we will also put $M' = \lfloor (F_n - 1)/2 \rfloor$. We write $c_{nt} = s_{n0}/s_{nt}$ and $c_{\infty t} = \left(2\sqrt{5}t(1 - \frac{\xi_{\infty t}}{\sqrt{5}t})\right)^{-1}$ so that (5.4) becomes:

$$\left(\prod_{t=1}^{M'} \frac{1 - c_{nt}^2}{1 - c_{\infty t}^2}\right) \Big/ \prod_{t=M'+1}^{\infty} (1 - c_{\infty t}^2) \geq C_n/C_\infty \geq \left(\prod_{t=1}^{M} \frac{1 - c_{nt}^2}{1 - c_{\infty t}^2}\right) \Big/ \prod_{t=M+1}^{\infty} (1 - c_{\infty t}^2) \tag{5.35}$$

Analysis of the denominator Recall $|\xi_{\infty t}| = |\{t\omega\} - 1/2| < 1/2$ for $t \geq 1$ and so:

$$1 > \frac{1}{\left(2\sqrt{5}t - 1\right)} > c_{\infty t} > \frac{1}{\left(2\sqrt{5}t + 1\right)} > 0 \tag{5.36}$$

Hence $\sum_{N+1}^{\infty} c_{\infty t}^2 < \sum_{N+1}^{\infty} \left(2\sqrt{5}t - 1\right)^{-2} < \frac{1}{20} \int_N^\infty (t - 1/2\sqrt{5})^{-2} < 1/20(N - 1/2\sqrt{5})$ which is less than 1 for any $N \geq 1$. We can therefore use the additive version of the Product Inequality (5.9) in the denominator of (5.35) to obtain for any $N \geq 1$:

$$1 < \frac{1}{\prod_{t=N+1}^{\infty}(1 - c_{\infty t}^2)} < \frac{1}{1 - \sum_{N+1}^{\infty} c_{\infty t}^2} < \frac{1}{1 - \frac{1}{20\left(N - \frac{1}{2\sqrt{5}}\right)}} \tag{5.37}$$

Note that the upper bound decreases monotonically to 1 with N.

Analysis of the numerator Note that for $N \in \{M, M'\}$, the numerators in (5.35) can be written as:

$$\prod_{t=1}^{N} \frac{1 - c_{nt}^2}{1 - c_{\infty t}^2} = \prod_{t=1}^{N} 1 - \frac{c_{nt}^2 - c_{\infty t}^2}{1 - c_{\infty t}^2} \tag{5.38}$$

We will use the exponential Product Inequality to provide an upper bound for this product, but this becomes extremely unwieldy to use for the lower bound. Instead we will use the additive Product Inequality (5.9) to give a lower bound for the RHS, but to do this we will first need to show that $\sum_{t=1}^{M} \frac{|c_{nt}^2 - c_{\infty t}^2|}{1 - c_{\infty t}^2} < 1$.

We proceed to analyse the terms $c_{nt}^2 - c_{\infty t}^2$. Since both $c_{nt}, c_{\infty t}$ are positive we can write $\left|c_{nt}^2 - c_{\infty t}^2\right| = (c_{nt} + c_{\infty t}) \left|c_{nt} - c_{\infty t}\right|$, and we proceed to analyse the terms on the RHS. First we examine $c_{nt} \pm c_{\infty t}$:

$$
\begin{aligned}
c_{nt} \pm c_{\infty t} &= \frac{\sin \pi\omega^n/2}{\sin \pi(t/F_n - \omega^n \xi_{nt})} \pm \frac{1}{2\sqrt{5}t\left(1 - \xi_{\infty t}/\sqrt{5}t\right)} \\
&= \frac{2\sqrt{5}t\,(\sin \pi\omega^n/2)\left(1 - \xi_{\infty t}/\sqrt{5}t\right) \pm \sin \pi(t/F_n - \omega^n \xi_{nt})}{2\sqrt{5}t \sin \pi(t/F_n - \omega^n \xi_{nt})\left(1 - \xi_{\infty t}/\sqrt{5}t\right)} \qquad (5.39)
\end{aligned}
$$

Note that $t/F_n - \omega^n \xi_{nt} = (t/F_n)(1 - F_n\omega^n \xi_{nt}/t)$ and put $\gamma_{nt} = 1 - F_n\omega^n\xi_{nt}/t = 1 - (1 - (-1)^n\omega^{2n})\xi_{nt}/\sqrt{5}t$. We proceed to analyse the numerators for each case, beginning with the numerator $\mathfrak{N}^+$ of $c_{nt} + c_{\infty t}$:

$$
\begin{aligned}
\mathfrak{N}^+(t) &= 2\sqrt{5}t\,(\sin \pi\omega^n/2)\left(1 - \xi_{\infty t}/\sqrt{5}t\right) + \sin \pi(t/F_n - \omega^n \xi_{nt}) \\
&< 2\sqrt{5}t\,(\pi\omega^n/2)\left(1 - \xi_{\infty t}/\sqrt{5}t\right) + \frac{\pi t}{F_n}\gamma_{nt} \\
&< \frac{\pi t}{F_n}\left(\sqrt{5}F_n\omega^n\left(1 + \frac{1}{2\sqrt{5}t}\right) + 1 + \frac{(1+\omega^{2n})}{2\sqrt{5}t}\right) \\
&< \frac{\pi t}{F_n}\left((1+\omega^{2n})\left(1 + \frac{1}{2\sqrt{5}t}\right) + 1 + \frac{1+\omega^{2n}}{2\sqrt{5}t}\right) \\
&< \frac{\pi t}{F_n}\left(2 + \omega^{2n} + \frac{(1+\omega^{2n})}{\sqrt{5}t}\right) \qquad (5.40)
\end{aligned}
$$

We now examine the numerator $\mathfrak{N}^-$ of $c_{nt} - c_{\infty t}$ (using $\xi_{nt} = \xi_{\infty t} + t(-\omega)^n/F_n$):

$$
\begin{aligned}
\mathfrak{N}^-(t) &= 2\sqrt{5}t\,(\sin \pi\omega^n/2)\left(1 - \xi_{\infty t}/\sqrt{5}t\right) - \sin \pi(t/F_n - \omega^n \xi_{nt}) \\
&< 2\sqrt{5}t\,(\pi\omega^n/2)\left(1 - \xi_{\infty t}/\sqrt{5}t\right) - \frac{\pi t}{F_n}\gamma_{nt}\left(1 - \frac{1}{6}\left(\frac{\pi t}{F_n}\gamma_{nt}\right)^2\right) \\
&= \frac{\pi t}{F_n}\left(\sqrt{5}F_n\omega^n\left(1 - \frac{\xi_{\infty t}}{\sqrt{5}t}\right) - \left(1 - \frac{\xi_{nt}}{\sqrt{5}t}\left(1 - (-1)^n\omega^{2n}\right)\right)\left(1 - \frac{1}{6}\left(\frac{\pi t}{F_n}\gamma_{nt}\right)^2\right)\right) \\
&= \frac{\pi t}{F_n}\left((1 - (-1)^n\omega^{2n})\left(1 - \frac{\xi_{\infty t}}{\sqrt{5}t}\right) - \left(1 - \frac{\xi_{nt}}{\sqrt{5}t}\left(1 - (-1)^n\omega^{2n}\right)\right)\left(1 - \frac{1}{6}\left(\frac{\pi t}{F_n}\gamma_{nt}\right)^2\right)\right) \\
&= \frac{\pi t}{F_n}\left(\frac{\xi_{nt} - \xi_{\infty t}}{\sqrt{5}t} - (-1)^n\omega^{2n}\left(1 - \frac{\xi_{\infty t} - \xi_{nt}}{\sqrt{5}t}\right) + \gamma_{nt}\frac{1}{6}\left(\frac{\pi t}{F_n}\gamma_{nt}\right)^2\right) \\
&= \frac{\pi t}{F_n}\left(\frac{(-\omega)^n}{\sqrt{5}F_n} - (-1)^n\omega^{2n}\left(1 + \frac{(-\omega)^n}{\sqrt{5}F_n}\right) + \frac{1}{6}\left(\frac{\pi t}{F_n}\right)^2\gamma_{nt}^3\right) \\
&< \frac{\pi t}{F_n}\left(\frac{\omega^n(1+\omega^{2n})}{\sqrt{5}F_n} + \omega^{2n} + \frac{1}{6}\left(\frac{\pi t}{F_n}\right)^2\left(1 + \frac{1}{2\sqrt{5}t}\left(1 + \omega^{2n}\right)\right)^3\right)
\end{aligned}
$$

This gives us an upper bound, but since $c_{nt} - c_{\infty t}$ may be negative we also need a lower bound.

Again we use $\xi_{nt} = \xi_{\infty t} + t(-\omega)^n/F_n)$ in the numerator $\mathfrak{N}^-$ to obtain:

$$\begin{aligned}
\mathfrak{N}^-(t) &= 2\sqrt{5}t\,(\sin \pi\omega^n/2)\left(1-\xi_{\infty t}/\sqrt{5}t\right) - \sin\pi(t/F_n - \omega^n\xi_{nt}) \\
&> 2\sqrt{5}t\,(\pi\omega^n/2)\left(1-\frac{1}{6}\left(\frac{\pi\omega^n}{2}\right)^2\right)\left(1-\xi_{\infty t}/\sqrt{5}t\right) - \frac{\pi t}{F_n}\gamma_{nt} \\
&= \frac{\pi t}{F_n}\left(\sqrt{5}F_n\omega^n\left(1-\frac{1}{6}\left(\frac{\pi\omega^n}{2}\right)^2\right)\left(1-\xi_{\infty t}/\sqrt{5}t\right) - \gamma_{nt}\right) \\
&= \frac{\pi t}{F_n}\left((1-(-1)^n\omega^{2n})\left(1-\frac{1}{6}\left(\frac{\pi\omega^n}{2}\right)^2\right)\left(1-\xi_{\infty t}/\sqrt{5}t\right) - \left(1-\frac{\xi_{nt}}{\sqrt{5}t}\left(1-(-1)^n\omega^{2n}\right)\right)\right) \\
&= \frac{\pi t}{F_n}\left(\frac{\xi_{nt}-\xi_{\infty t}}{\sqrt{5}t}\left(1-(-1)^n\omega^{2n}\right) + \frac{1}{6}\left(\frac{\pi\omega^n}{2}\right)^2\frac{\xi_{\infty t}}{\sqrt{5}t}\left(1-(-1)^n\omega^{2n}\right)\right. \\
&\quad \left. - \left(1-(-1)^n\omega^{2n}\right)\frac{1}{6}\left(\frac{\pi\omega^n}{2}\right)^2 - (-1)^n\omega^{2n}\right) \\
&> -\frac{\pi t}{F_n}\left(\frac{\omega^n(1+\omega^{2n})}{\sqrt{5}F_n} + \left(\frac{\pi^2\omega^{2n}}{24}\right)\frac{1}{2\sqrt{5}t}\left(1+\omega^{2n}\right) + \left(1+\omega^{2n}\right)\left(\frac{\pi^2\omega^{2n}}{24}\right) + \omega^{2n}\right) \\
&= -\frac{\pi t}{F_n}\left(\frac{\omega^n(1+\omega^{2n})}{\sqrt{5}F_n} + \left(1+\omega^{2n}\right)\left(\frac{\pi^2\omega^{2n}}{24}\right)\left(1+\frac{1}{2\sqrt{5}t}\right) + \omega^{2n}\right)
\end{aligned}$$

Hence by forming the superset of terms from both upper and lower bounds:

$$\begin{aligned}
|\mathfrak{N}^-(t)| &< \frac{\pi t}{F_n}\left(\frac{\omega^n(1+\omega^{2n})}{\sqrt{5}F_n} + \omega^{2n} + \left(1+\omega^{2n}\right)\left(\frac{\pi^2\omega^{2n}}{24}\right)\left(1+\frac{1}{2\sqrt{5}t}\right)\right. \\
&\quad \left. + \frac{1}{6}\left(\frac{\pi t}{F_n}\right)^2\left(1+\frac{1}{2\sqrt{5}t}\left(1+\omega^{2n}\right)\right)^3\right)
\end{aligned} \tag{5.41}$$

Finally using (5.39) we get $\left|c_{nt}^2 - c_{\infty t}^2\right| = (c_{nt} + c_{\infty t})\,|c_{nt} - c_{\infty t}| = (\mathfrak{N}^+/\mathfrak{D})\,(|\mathfrak{N}^-|/\mathfrak{D})$ where $\mathfrak{D} = \left(2\sqrt{5}t(\sin\frac{\pi t}{F_n}\gamma_{nt})\left(1-\xi_{\infty t}/\sqrt{5}t\right)\right)$ and hence:

$$\begin{aligned}
\left|c_{nt}^2 - c_{\infty t}^2\right| &= \frac{\mathfrak{N}^+(t)\,|\mathfrak{N}^-(t)|}{\left(2\sqrt{5}t(\sin\frac{\pi t}{F_n}\gamma_{nt})\left(1-\xi_{\infty t}/\sqrt{5}t\right)\right)^2} \\
&< \frac{\mathfrak{N}^+\,|\mathfrak{N}^-|}{20t^2\left(\frac{\pi t}{F_n}\right)^2\gamma_{nt}^2\left(1-\frac{1}{6}\left(\frac{\pi t}{F_n}\gamma_{nt}\right)^2\right)^2\left(1-\xi_{\infty t}/\sqrt{5}t\right)^2} \\
&< \frac{\left(\frac{\pi t}{F_n}\right)^{-2}\mathfrak{N}^+\,|\mathfrak{N}^-|}{20t^2\left(1-\frac{1}{2\sqrt{5}t}\left(1+\omega^{2n}\right)\right)^2\left(1-\frac{1}{6}\left(\frac{\pi t}{F_n}\left(1+\frac{1}{2\sqrt{5}t}\left(1+\omega^{2n}\right)\right)\right)^2\right)^2\left(1-\frac{1}{2\sqrt{5}t}\right)^2}
\end{aligned} \tag{5.42}$$

Finally write $\mathfrak{N}(t)$ for the numerator of this fraction, so that (using (5.40), (5.41)) we obtain:

$$\begin{aligned}\mathfrak{N}(t) &= \left(\frac{\pi t}{F_n}\right)^{-2} \mathfrak{N}^+(t)\,|\mathfrak{N}^-(t)| \qquad (5.43)\\ &< \left(2+\omega^{2n}+\frac{(1+\omega^{2n})}{\sqrt{5}t}\right) \quad \times\end{aligned}$$

$$\left(\frac{\omega^n(1+\omega^{2n})}{\sqrt{5}F_n}+\omega^{2n}+\left(1+\omega^{2n}\right)\left(\frac{\pi^2\omega^{2n}}{24}\right)\left(1+\frac{1}{2\sqrt{5}t}\right)+\frac{1}{6}\left(\frac{\pi t}{F_n}\right)^2\left(1+\frac{1}{2\sqrt{5}t}\left(1+\omega^{2n}\right)\right)^3\right)$$

Writing $N = \lceil \omega^{-n/2} \rceil$ we now consider the instances of (5.42) in which $M \geq t > N > \omega^{-n/2}$. We note that this requires $M \geq N+1$, requiring $\lfloor F_n/2 \rfloor \geq \lceil \omega^{-n/2} \rceil + 1$ which in turn requires $n \geq 8$. Now using $F_n/2 > t > \omega^{-n/2}$ in (5.42) and (5.43) we obtain:

$$\sum_{t=N+1}^{M} \left|c_{nt}^2 - c_{\infty t}^2\right| <$$

$$\sum_{t=N+1}^{M}\left(\frac{\left(2+\omega^{2n}+\frac{(1+\omega^{2n})\omega^{n/2}}{\sqrt{5}}\right)}{20t^2\left(1-\frac{\omega^{n/2}}{2\sqrt{5}}\left(1+\omega^{2n}\right)\right)^2\left(1-\frac{1}{6}\left(\frac{\pi}{2}\left(1+\frac{\omega^{n/2}}{2\sqrt{5}}\left(1+\omega^{2n}\right)\right)\right)^2\right)^2\left(1-\frac{\omega^{n/2}}{2\sqrt{5}}\right)^2}\quad\times\right.$$

$$\left.\left(\frac{\omega^n(1+\omega^{2n})}{\sqrt{5}F_n}+\omega^{2n}+\left(1+\omega^{2n}\right)\left(\frac{\pi^2\omega^{2n}}{24}\right)\left(1+\frac{\omega^{n/2}}{2\sqrt{5}}\right)+\frac{1}{6}\left(\frac{\pi}{2}\right)^2\left(1+\frac{\omega^{n/2}}{2\sqrt{5}}\left(1+\omega^{2n}\right)\right)^3\right)\right)$$

$$= \sum_{N+1}^{M} \frac{K_1(n)}{t^2} < \int_N^\infty \frac{K_1}{t^2} = \frac{K_1}{N} < K_1\omega^{n/2}$$

Note that each term in the numerator of $K_1(n)$ decreases with n, whereas each term in the denominator increases, so that $K_1(n)$ decreases monotonically with n and hence so does the sum $\sum_{N+1}^{M}\left|c_{nt}^2 - c_{\infty t}^2\right|$. By direct calculation we obtain:

$$K_1(8) < 0.170$$

We now consider the remaining instances of (5.42), namely those in which $1 \leq t \leq N$. But in these instances $t/F_n < \lceil \omega^{-n/2} \rceil \sqrt{5}/\omega^{-n}(1-\omega^{2n}) < (1+\omega^{-n/2})\sqrt{5}\omega^n/(1-\omega^{2n})$ giving us:

$$\sum_1^N \left|c_{nt}^2 - c_{\infty t}^2\right| <$$

$$\sum_1^N \left(\frac{\left(2+\omega^{2n}+\frac{(1+\omega^{2n})}{\sqrt{5}}\right)}{20t^2\left(1-\frac{1}{2\sqrt{5}}\left(1+\omega^{2n}\right)\right)^2\left(1-\frac{5\pi^2\omega^{2n}(1+\omega^{-n/2})^2}{6(1-\omega^{2n})^2}\left(1+\frac{1}{2\sqrt{5}}\left(1+\omega^{2n}\right)\right)^2\right)^2\left(1-\frac{1}{2\sqrt{5}}\right)^2} \times \right.$$

$$\left(\frac{\omega^n(1+\omega^{2n})}{\sqrt{5}F_n} + \omega^{2n} + \left(1+\omega^{2n}\right)\left(\frac{\pi^2\omega^{2n}}{24}\right)\left(1+\frac{1}{2\sqrt{5}}\right) \right.$$

$$\left.\left. + \frac{5\pi^2\omega^{2n}(1+\omega^{-n/2})^2}{6(1-\omega^{2n})^2}\left(1+\frac{1}{2\sqrt{5}}\left(1+\omega^{2n}\right)\right)^3 \right)\right)$$

$$= \sum_1^N \frac{K_2(n)}{t^2} < K_2\frac{\pi^2}{6}$$

Once again, inspection of the terms shows that $K_2(n)$ is monotonically decreasing with n for $n \geq 0$. By direct calculation we obtain:

$$K_2(8) < 0.280$$

Now note from (5.36) that $1 - c_{\infty t}^2 \geq 1 - c_{\infty 1}^2$ and so for $n \geq 8$:

$$\sum_{t=1}^M \frac{\left|c_{nt}^2 - c_{\infty t}^2\right|}{1 - c_{\infty t}^2} \leq \frac{1}{1 - c_{\infty 1}^2}\sum_{t=1}^M \left|c_{nt}^2 - c_{\infty t}^2\right| < \frac{K_1(n)\omega^{n/2} + K_2(n)\frac{\pi^2}{6}}{1 - c_{\infty 1}^2} \tag{5.44}$$

Note that the term on the RHS is again monotonically decreasing with n, and again by direct calculation (using $c_{\infty 1} > 1 - \left(\frac{1}{(2\sqrt{5}-1)}\right)^2 \approx 0.917$) it is easily verified to be less than unity for $n \geq 8$ so that we are now finally justified in using the additive version of the Product Inequality (5.9).

Final results For $n \geq 8$, we can now use inequalities (5.11), (5.9) in (5.35) to obtain:

$$\exp\left(\sum_{t=1}^{M'} \frac{\left|c_{nt}^2 - c_{\infty t}^2\right|}{1 - c_{\infty t}^2}\right) \Bigg/ \prod_{t=M'+1}^{\infty} (1 - c_{\infty t}^2) > \frac{C_n}{C_\infty} > \left(1 - \sum_{t=1}^{M} \frac{\left|c_{nt}^2 - c_{\infty t}^2\right|}{1 - c_{\infty t}^2}\right) \Bigg/ \prod_{t=M+1}^{\infty} (1 - c_{\infty t}^2) \tag{5.45}$$

We can now use the estimate (5.37):

$$\exp\left(\sum_{t=1}^{M'} \frac{\left|c_{nt}^2 - c_{\infty t}^2\right|}{1 - c_{\infty t}^2}\right) \frac{1}{1 - \frac{1}{20\left(M' - \frac{1}{2\sqrt{5}}\right)}} > \frac{C_n}{C_\infty} > \left(1 - \sum_{t=1}^{M} \frac{\left|c_{nt}^2 - c_{\infty t}^2\right|}{1 - c_{\infty t}^2}\right) \tag{5.46}$$

Finally using (5.44) gives for $n \geq 8$:

$$\exp\left(\frac{K_1(n)\omega^{n/2} + K_2(n)\frac{\pi^2}{6}}{1 - c_{\infty 1}^2}\right)\frac{1}{1 - \frac{1}{20\left(M' - \frac{1}{2\sqrt{5}}\right)}} > \frac{C_n}{C_\infty} > \left(1 - \frac{K_1(n)\omega^{n/2} + K_2(n)\frac{\pi^2}{6}}{1 - c_{\infty 1}^2}\right) \tag{5.47}$$

Once again this bracket converges monotonically to 1 with n.

5.3.4.5 Final results on the bounds for P_{F_n}

n	$P_{F_n} \approx$
0	1.00000
1	1.86406
2	1.86406
3	2.51832
4	2.22856
5	2.48203
6	2.34584
7	2.43962
8	2.38489
9	2.42005
10	2.39881
11	2.41213
12	2.40397
13	2.40904
14	2.40592
15	2.40785
16	2.40666
17	2.40740
18	2.40694
19	2.40722
20	2.40705

Table 5.3.1: P_{F_n} for $n \leq 20$

At this point we have established brackets for each of the ratios $(A_n/A_\infty), (B_n/B_\infty), (C_n/C_\infty)$, valid for $n \geq 8$ and monotonically converging to 1 with n. Now $A_nB_nC_n = P_{F_n}$and by the main result of the chapter 4, we have $P_{F_n} \to c = A_\infty B_\infty C_\infty$. It follows that for each n we can multiply together the brackets for $(A_n/A_\infty), (B_n/B_\infty), (C_n/C_\infty)$ to obtain a bracket $x_n < P_{F_n}/c < y_n$, and further that the brackets $[x_n, y_n]$ are also monotonically converging to 1 with n.

We will use this information to establish the necessary lower bound for P_{F_n} to establish the Fibonacci Hypothesis and hence the lower bound on P_k. To do this we will need to make some numerical calculations using our formulae, and the results are set out in the tables of this section. We computed the tables using Java and 64 bit floating point operations (15-17 digits of precision) as this was sufficient for our purposes. The precision of each individual result is highly dependent of the size of the Fibonacci numbers involved: the largest one we have used is $F_{32} = 2178309$ (7 digits) which gives an estimated worst precision of 5 digits in the results; smaller Fibonacci numbers give much better precision, for example $F_{20} = 6765$ (4 digits) gives an estimated 8-9 digits precision in the results. In the tables themselves, the computed values have been rounded up for upper bounds, down for lower bounds, and to the nearest decimal place for actuals.

We now turn to the calculations. First, since $x_n < P_{F_n}/c < y_n$, we have $P_{F_n} > cx_n$. But we also have $P_{F_N}/y_N < c$ giving us $P_{F_n} > x_nP_{F_N}/y_N$. If we fix N, then since x_n is monotonically increasing, it follows that $x_nP_{F_N}/y_N$ is a sequence of monotonically increasing lower bounds for P_{F_n}. Similarly $x_NP_{F_N}/y_n$ is a sequence of monotonically decreasing upper bounds for P_{F_n}. Finally since $P_{F_n} \to c$, these upper and lower bounds are also a bracket for c.

We can now calculate the brackets for P_{F_n}/c, and the results are shown in tables 5.3.2 and 5.3.3. Setting $N = 32$, in table 5.3.4 we then show the brackets calculated for P_{F_n} surrounding the actual (directly calculated) values of P_{F_n}. From this latter table (and recalling that sequences of bounds are monotonic) we can read off $P_{F_n} > 1.792 > P$ for $n \geq 20$, where $P \approx 1.781$ is the crucial bound required to establish the Fibonacci Hypothesis calculated in (5.2).

It remains to examine the values of P_{F_n} for $n < 20$. We tabulate the calculated actual values of P_{F_n} in Table 5.3.1 which show $P_{F_n} > 1.864 > P$ for $20 \geq n \geq 1$. Hence $P_{F_n} > 1.792 > P$ for all

n	$A_n/A_\infty >$	$B_n/B_\infty >$	$C_n/C_\infty >$	$P_{F_n}/c > x_n =$
8	0.998	0.038	0.471	0.018
9	0.999	0.074	0.784	0.058
10	0.999	0.132	0.890	0.117
11	0.999	0.190	0.938	0.178
12	0.999	0.243	0.963	0.234
13	0.999	0.318	0.977	0.311
14	0.999	0.401	0.985	0.395
15	0.999	0.466	0.990	0.462
16	0.999	0.537	0.993	0.534
17	0.999	0.604	0.995	0.601
18	0.999	0.665	0.996	0.663
19	0.999	0.716	0.997	0.714
20	0.999	0.761	0.998	0.760
21	0.999	0.802	0.998	0.801
22	0.999	0.836	0.999	0.835
23	0.999	0.865	0.999	0.864
24	0.999	0.888	0.999	0.888
25	0.999	0.909	0.999	0.908
26	0.999	0.925	0.999	0.925
27	0.999	0.939	0.999	0.939
28	0.999	0.950	0.999	0.950
29	0.999	0.960	0.999	0.960
30	0.999	0.967	0.999	0.967
31	0.999	0.973	0.999	0.973
32	0.999	0.978	0.999	0.978

Table 5.3.2: Lower bounds x_n for P_{F_n}/c

$n \geq 1$ which establishes the Fibonacci Hypothesis (5.3.3), namely that for $y \in I = [-\omega, \omega^2]$ and $n \geq 2$, we have $P_{F_n}((-\omega)^n y) > 1$.

In turn this establishes lemma 5.3.4, namely that or any $k \geq 0$ with Fibonacci index $\iota(k)$ (so that $F_{\iota(k)} \leq k < F_{\iota(k)+1}$), we have:

$$P_k((-\omega)^{\iota(k)}y) \geq P_{F_{\iota(k)}}((-\omega)^{\iota(k)}y) \geq 1$$

with the first equality holding only for $k = F_{\iota(k)}$ and second equality only for $k = F_0 = 0$.

5.4 Proof of the upper bound $P_k < Ck$

The hard work is now done, and we can now use the bounds we have obtained in section 5.3 to derive an upper bound on the growth rate of P_k.

We start by using the lower bound for $P_k((-\omega)^n y)$ to establish an upper bound in terms of $P_{F_n-1}((-\omega)^n y)$.

Lemma 5.4.1. *For $n \geq 3$, $F_{n-1} \leq k < F_n$ and $y \in J = [-\omega^{-1}, \omega^{-2}]$ we have $P_k((-\omega)^n y) \leq P_{F_n-1}((-\omega)^n y)$ with equality only for $k = F_n - 1$.*

n	$A_n/A_\infty <$	$B_n/B_\infty <$	$C_n/C_\infty <$	$P_{F_n}/c < y_n =$
8	1.001	25.861	1.706	44.133
9	1.001	13.409	1.245	16.692
10	1.001	7.559	1.118	8.447
11	1.001	5.258	1.065	5.598
12	1.001	4.100	1.039	4.256
13	1.001	3.138	1.024	3.212
14	1.001	2.489	1.016	2.527
15	1.001	2.142	1.010	2.163
16	1.001	1.861	1.007	1.873
17	1.001	1.656	1.005	1.663
18	1.001	1.503	1.004	1.507
19	1.001	1.397	1.003	1.400
20	1.001	1.313	1.002	1.315
21	1.001	1.247	1.002	1.248
22	1.001	1.196	1.001	1.197
23	1.001	1.156	1.001	1.157
24	1.001	1.126	1.001	1.126
25	1.001	1.100	1.001	1.101
26	1.001	1.081	1.001	1.081
27	1.001	1.065	1.001	1.065
28	1.001	1.052	1.001	1.052
29	1.001	1.042	1.001	1.042
30	1.001	1.034	1.001	1.034
31	1.001	1.027	1.001	1.027
32	1.001	1.022	1.001	1.022

Table 5.3.3: Upper bounds y_n for P_{F_n}/c

n	$P_{F_n} >$	$P_{F_n} \approx$	$P_{F_n} <$
8	0.042	2.385	108.514
9	0.137	2.420	41.041
10	0.277	2.399	20.768
11	0.420	2.412	13.762
12	0.553	2.404	10.465
13	0.733	2.409	7.896
14	0.932	2.406	6.212
15	1.089	2.408	5.318
16	1.258	2.407	4.605
17	1.417	2.407	4.088
18	1.563	2.407	3.706
19	1.683	2.407	3.441
20	1.792	2.407	3.233
21	1.887	2.407	3.068
22	1.968	2.407	2.942
23	2.036	2.407	2.845
24	2.092	2.407	2.768
25	2.140	2.407	2.706
26	2.180	2.407	2.657
27	2.212	2.407	2.618
28	2.239	2.407	2.587
29	2.261	2.407	2.562
30	2.279	2.407	2.542
31	2.293	2.407	2.525
32	2.305	2.407	2.513

Table 5.3.4: Brackets (using $N = 32$) and actuals for P_{F_n}

Proof. By definition we have

$$P_k((-\omega)^n y) = P_{F_n-1}((-\omega)^n y) \Big/ \left|\prod_{r=k+1}^{F_n-1} 2\sin\pi(r\omega + (-\omega)^n y)\right| \tag{5.48}$$

Putting $s = F_n - r$ in the right-hand product (and using $\sin(-x) = -\sin x$) gives:

$$\begin{aligned}\left|\prod_{r=k+1}^{F_n-1} 2\sin\pi(r\omega + (-\omega)^n y)\right| &= \left|\prod_{s=1}^{F_n-1-k} 2\sin\pi((F_n - s)\omega + (-\omega)^n y)\right| \\ &= \left|\prod_{s=1}^{F_n-1-k} 2\sin\pi(s\omega + (-\omega)^n - (-\omega)^n y)\right| \\ &= P_{F_n-1-k}((-\omega)^n(1-y)) \end{aligned} \tag{5.49}$$

Now $P_{F_n-1-k}((-\omega)^n(1-y)) = P_{k'}((-\omega)^m x)$ where $x = (-\omega)^{n-m}(1-y)$, $k' = F_n - 1 - k$ and m is the Fibonacci index of k' so that $F_m \le k' < F_{m+1}$. Since $k \ge F_{n-1}$ we have $k' \le F_{n-2} - 1$ so that $m \le n-3$ (which requires $n \ge 3$). We can now use the lower bound result from lemma 5.3.4 to give us for $x \in I = [-\omega, \omega^2]$ and Fibonacci index $m \ge 0$:

$$P_{F_n-1-k}((-\omega)^m x) \ge P_{F_m}((-\omega)^m x) \ge 1 \tag{5.50}$$

with the second equality holding only for $m = 0$, which in turn requires $F_n - 1 - k = 0$, ie $k = F_n - 1$.

Now using $\omega^{-1} = 1 + \omega, \omega^{-2} = 2 + \omega$, we obtain $(-\omega)^3 J = [(-\omega)^3(-\omega^{-1}), (-\omega)^3\omega^{-2}] = [\omega^2, -\omega] = I$ and $1 - J = [1 - (-\omega)^{-1}, 1 - \omega^{-2}] = [2 + \omega, -1 - \omega] = J$. Hence if $y \in J$ (and $n \ge 3$) we have $x = (-\omega)^{n-m}(1-y) \in (-\omega)^{n-m-3}(-\omega)^3(1-J) = (-\omega)^{n-m-3}I \subseteq I$, ie $x \in I$. Hence (5.50) gives us $P_{F_n-1-k}((-\omega)^n(1-y)) \ge 1$ for $y \in J$ (with equality only for $k = F_n - 1$). Combining (5.48) and (5.49) now gives us for $y \in J$ and $n \ge 3$:

$$P_k((-\omega)^n y) = P_{F_n-1}((-\omega)^n y) / P_{F_n-1-k}((-\omega)^n(1-y)) \le P_{F_n-1}((-\omega)^n y)$$

with final equality only for $k = F_n - 1$. □

Corollary 5.4.2. *For $n \ge 1$ and $F_{n-1} \le k < F_n$ the Sudler product $P_k = P_k(0)$ satisfies $P_k \le P_{F_n-1}$ with equality only for $k = F_n - 1$*

Proof. Noting $0 \in J$, the lemma gives us the result putting $y = 0$ and $n \ge 3$. It remains to consider the cases $n = 1, 2$. But then the conditions $k < F_1 = F_2 = 1$ require $k = 0$ and also $F_n - 1 = 0$, so that for $n = 1, 2$, $k = F_n - 1 = 0$ and $P_k = 1 = P_{F_n-1}$. □

We can now proceed to prove the main result of this section, namely that the Sudler product

P_k has linear growth.

Lemma 5.4.3. *P_k has at most linear growth (there is a C such that for any $k \geq 1$ we have $P_k < Ck$) and the subsequence P_{F_n-1} has at least linear growth.*

Proof. We start by noting that for $n \geq 1$:

$$P_{F_n-1} = P_{F_n} / |2 \sin \pi(-(-\omega)^n)| \tag{5.51}$$

and hence for $n \geq 1$:

$$\frac{P_{F_n}}{2\pi\omega^n} < P_{F_n-1} < \frac{P_{F_n}}{2\pi\omega^n(1 - \frac{1}{6}\pi^2\omega^{2n})} \tag{5.52}$$

For $n \geq 0$ we use $F_n = \frac{1}{\sqrt{5}}(\omega^{-n} - (-\omega)^n)$ to obtain $\omega^{-n} = \sqrt{5}F_n/(1 - (-1)^n\omega^{2n})$ which we use in the LHS to obtain:

$$\begin{aligned} P_{F_n-1} &> \frac{\sqrt{5}P_{F_n}}{2\pi(1+\omega^{2n})} F_n \\ &> \frac{\sqrt{5}P_{F_n}}{2\pi(1+\omega^{2n})} (F_n - 1) \end{aligned} \tag{5.53}$$

From tables 5.3.1,5.3.4 we can see that $P_{F_n} \geq 1$ for $n \geq 0$, and so the subsequence P_{F_n-1} has at least linear growth, which establishes the second part of the lemma.

For $n \geq 1$ we use $F_{n-1} = \frac{1}{\sqrt{5}}(\omega^{-(n-1)} - (-\omega)^{n-1})$ to obtain $\omega^{-(n-1)} = \sqrt{5}F_{n-1}/(1-(-1)^{n-1}\omega^{2n-2})$ for $n \geq 2$ (we must discard the case $n = 1$ as the denominator becomes 0). We use the result in the RHS of (5.52) to obtain for $n \geq 2$:

$$P_{F_n-1} < \frac{\sqrt{5}P_{F_n}}{2\pi\omega(1 - \omega^{2n-2})(1 - \frac{1}{6}\pi^2\omega^{2n})} F_{n-1}$$

Now by corollary 5.4.2, for each $n \geq 1$ and any $F_{n-1} \leq k < F_n$, we have $P_k \leq P_{F_n-1}$, and so for $n \geq 2$:

$$P_k \leq P_{F_n-1} < \frac{\sqrt{5}P_{F_n}}{2\pi\omega(1 - \omega^{2n-2})(1 - \frac{1}{6}\pi^2\omega^{2n})} k \tag{5.54}$$

From table 5.3.1 we see that $P_{F_n} < 2.519$ for $n \leq 20$, and from table 5.3.4 we see from the actuals column that $P_{F_n} < 2.421$ for $8 \leq n \leq 32$, and from the upper bound column that $P_{F_n} < 2.513$ for $n > 32$ (using the fact that the sequences of bounds are monotonic). Hence we can deduce that $P_{F_n} < 2.519$ for all n and the value $P_{F_N} = \max_{n \geq 0} P_{F_n} < 2.519$ exists and is achieved for $n = 3$. Now for $k \geq 1$, k has a Fibonacci floor of F_m with $m \geq 2$, which means k satisfies $F_{n-1} \leq k < F_n$ for some $n \geq 3$, and so from (5.54) we have:

$$P_k < \frac{\sqrt{5}P_{F_N}}{2\pi\omega(1 - \omega^2)(1 - \frac{1}{6}\pi^2\omega^6)} k \tag{5.55}$$

which establishes the first part of the lemma with (using $1 - \omega^2 = \omega$):

$$C = \sqrt{5} P_{F_N} / 2\pi\omega^2 (1 - \frac{1}{6}\pi^2\omega^6) \tag{5.56}$$

. □

Chapter 6

Fixed points of composition sum operators

In previous chapters we have developed a series of results in which renormalisation played an increasingly important role. In this chapter we go a step further by developing a general renormalisation scheme. This proves a very complex scheme so we will focus on the simplest representative, which is the golden renormalisation operator M defined on functions of the complex plane by:

$$(Mf)(z) = f(-\omega z) + f(\omega^2 z + \omega) \tag{6.1}$$

where ω is the golden rotation $\frac{1}{2}\left(\sqrt{5}-1\right)$. This operator arises in a number of contexts studied in the literature (see 6.2). We position this operator within what seems its natural context, that of composition sum operators (CSOs).

We will develop the theory of the fixed points of CSOs, and use it to determine constructively the complete set of fixed points of M in a suitable function space (see 6.5.2). This includes previously unknown fixed points, and surprisingly a fixed point which is not the limit of an infinite series. The construction also provides the new identity (6.33) as a byproduct.

The exposition in this chapter follows [58].

6.1 The general renormalisation scheme

In this section we will develop formally a procedure we have used several times to date. However this time, in addition to rescaling the variables we will also rescale the functions.

Let $S_n(x,\alpha) = \sum_{k=0}^{n-1} f(x + k\alpha)$ be a quasiperiodic sum for the irrational rotation α. (Note that the scheme we describe below is independent of the sum operator, and works just as well with the product operator). As usual we will let p_n/q_n be the convergents of α, and we will decimate the sequence by choosing every q_nth element. We now use the relationships $S_n(x,\alpha) = S_m(x,\alpha) + S_{n-m}(x+m\alpha,\alpha)$ from lemma 2.5.3, and $q_{n+1} = a_n q_n + q_{n-1}$ from (1.8), to derive the recurrence relation:

$$\begin{aligned} S_{q_{n+1}}(x,\alpha) &= S_{a_n q_n}(x,\alpha) + S_{q_{n-1}}(x + a_n q_n \alpha, \alpha) \\ &= \left(\sum_{k=0}^{a_n-1} S_{q_n}(x + k q_n \alpha, \alpha)\right) + S_{q_{n-1}}(x + a_n q_n \alpha, \alpha) \end{aligned}$$

Now by (1.6) $\{x + kq_n\alpha\} = \{x + k(p_n + (-1)^n\alpha_n)\} = \{x + k(-1)^n\alpha_n\}$ (where α_n is the absolute value of the error in the nth convergent), and so all the arguments in the expansion above are within $|k\alpha_n|$ of x. Now fix $x = x_0$ and put $\epsilon_n = (-1)^n\alpha_n$. We now rescale the coordinate system around x_0 by a factor of $1/\epsilon_n$ by putting $y = \{x - x_0\}/\epsilon_n$. This means the new co-ordinates of the circle are $y \in [0, 1/\epsilon_n)$ with the origin at the point of the circle with coordinate x_0 in the old system. (Note that ϵ_n is alternating.)

We now study the "renormalised" sum $\hat{S}_n(y) = S_{q_n}(x,\alpha)$. We note that $x = \{x_0 + \epsilon_n y\}$, so that:

$$\begin{aligned} \hat{S}_{n+1}(y) &= S_{q_{n+1}}(x_0 + \epsilon_{n+1}y, \alpha) \\ &= \sum_{k=0}^{a_n-1} S_{q_n}(x_0 + \epsilon_{n+1}y - k\epsilon_n, \alpha) + S_{q_{n-1}}(x_0 + \epsilon_{n+1}y - a_n\epsilon_n, \alpha) \\ &= \sum_{k=0}^{a_n-1} S_{q_n}\left(x_0 + \epsilon_n\left(\frac{\epsilon_{n+1}}{\epsilon_n}y - k\right), \alpha\right) + S_{q_{n-1}}\left(x_0 + \epsilon_{n-1}\left(\frac{\epsilon_{n+1}}{\epsilon_{n-1}}y - a_n\frac{\epsilon_n}{\epsilon_{n-1}}\right), \alpha\right) \\ &= \sum_{k=0}^{a_n-1} \hat{S}_n\left(\frac{\epsilon_{n+1}}{\epsilon_n}y - k\right) + \hat{S}_{n-1}\left(\frac{\epsilon_{n+1}}{\epsilon_{n-1}}y - a_n\frac{\epsilon_n}{\epsilon_{n-1}}\right) \end{aligned}$$

This functional recurrence relation is a representation of the renormalisation group operator. Its fixed points are the fixed points of the renormalisation (semi-) group. Unfortunately it is not a very simple recurrence relation. However in the special case of the golden ratio $\alpha = \omega = \frac{1}{2}\left(\sqrt{5} - 1\right)$ it does simplify considerably. In this case, $a_n = 1$ and $\epsilon_n/\epsilon_{n-1} = -\omega$ so that:

$$\hat{S}_{n+1}(y) = \hat{S}_n(-\omega y) + \hat{S}_{n-1}\left(\omega^2 y + \omega\right)$$

This particular functional recurrence appears in the golden ratio analysis of several important physical situations as we will see below. The crucial question in these situations is whether the series of functions $\hat{S}_n$ converges to a function $\hat{S}$. If it does then $\hat{S}$ is a fixed point of the operator M defined by:

$$(Mf)(y) = f(-\omega y) + f(\omega^2 y + \omega)$$

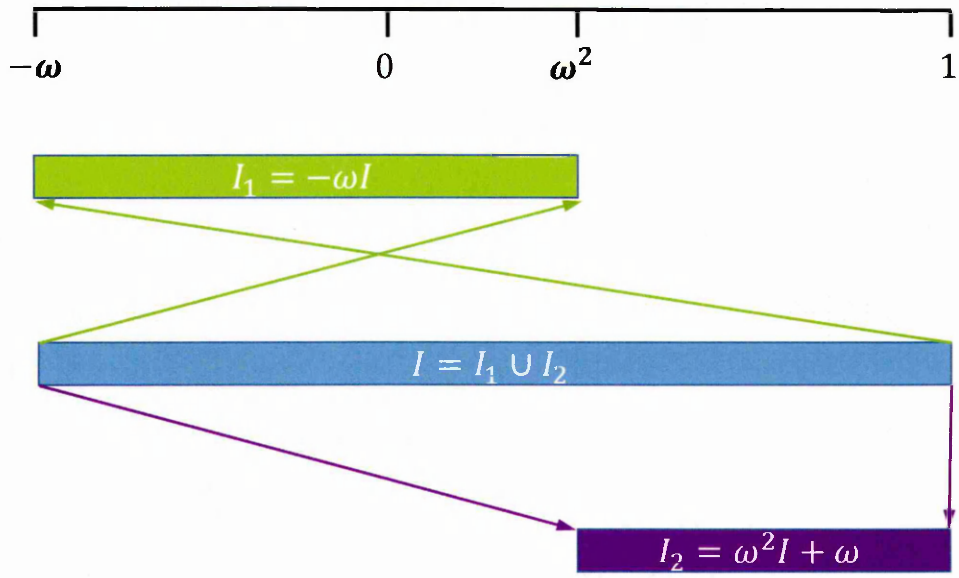

Figure 6.1.1: The attractor of the IFS defined by M is the interval I which is the self-affine union of the images of I under the two affine contractions of M

The operator M is clearly linear ($M(f_1 + f_2) = Mf_1 + Mf_2$), and bounded when f is bounded, so that at first sight our context seems to be the well trodden ground of bounded linear operators on a Banach space. Unfortunately it is not so simple. The physical situations giving rise to this problem lead us to be particularly interested in complex analytic solutions. We will show that M does indeed have a fixed point in Banach spaces of complex analytic functions, and that this fixed point is unique. Unfortunately this fixed point is the 0 function!

This leads immediately to the conclusion is that we will have to hunt for any non-trivial fixed points outside of Banach spaces. This lands us in the rather less well trod area of unbounded function spaces. We will need to develop a little of the theory of such spaces. However this turns out to be a good thing - once we realise for example that we can now expect a linear operator to have many fixed points, we realise that these fixed points can now form a non-trivial linear subspace, and there are some rich new algebraic structures which emerge. Of particular interest is the fact that there is a meta-operator '^' which maps a linear operator T to a fixed point operator $\hat{T}$ the image of which is precisely the fixed points of T.

First we review the relevant literature.

6.2 Review of existing results

A composition sum operator T is a linear operator acting on a vector space F of functions such that, for each function $f \in F$, the image vector Tf, evaluated at x, is given by:

$$Tf(x) = \sum_{i=1}^{\ell} a_i(x) f(\alpha_i(x)), \tag{6.2}$$

where $\ell \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are affine contractions, and a_1, a_2, $\ldots$, a_ℓ are a fixed sequence of coefficients which may in general be functions, although in many cases they will be constants. A full definition is given in section 6.3.2.

Composition sum operators have been studied extensively by Kuczma and his co-workers. In particular, in the seminal monograph [36], CSOs are discussed in detail in chapter 6 "Higher order equations and linear systems", principally in the real domain and in the "cyclic equation" case, in which the α_i are iterates of a single function. Kuczma *et al* give some important existence and uniqueness theorems in this context and we refer the reader to [36] and to the references contained therein. Functional equations associated with the operator (6.2) (and variations thereof) have been studied by several authors in the complex domain, following on from Koenigs' classical work [34] on the Schröder equation. Analytic solutions of a class of functional equations encompassing fixed points of (6.2) have been studied in particular by W. Smajdor [54], and K. Baron, R. Ger, J. Matkowski [3]. Investigations of analytic and meromorphic solutions of several functional equations of this general type have been undertaken by W. Smajdor, J. Matkowski, R. Goldstein, W. Pranger and R. Raclis. We refer the reader to [36] for an overview of this work and, in particular, to the references contained therein.

CSOs arise in several contexts, including the application of renormalisation techniques to quasiperiodic non-linear dynamical systems and a toy-model of magnetic flux growth in kinematic dynamo theory [18, 19].

Quasiperiodic systems are an important class of non-linear dynamical systems which find application in many areas of the physical sciences. In the simplest case, the dynamics are governed by an irrational number $\omega \notin \mathbb{Q}$, often called the rotation number or winding number in the literature. It can often be identified as the ratio of two incommensurate frequencies in the underlying system. Studies of quasiperiodic systems often focus on the time-correlations between system variables. These correlations, and, indeed other properties of quasiperiodic systems, typically depend on the number-theoretic properties of ω, and, in particular, on the continued-fraction expansion of ω and the associated rational convergents $p_n/q_n \to \omega$. Examples of quasiperiodic systems include strange non-chaotic attractors, the Harper equation and its generalisations, and other quantum mechanical models depending on an underlying irrational rotation of the circle.

The correlation structure of quasiperiodic systems may be understood by renormalisation analysis, leading to dynamical functional equations which relate correlations at time t to those at time $t + q_n$, the dynamical properties of which depend on the dynamical behaviour of the Gauss map ($x \mapsto x^{-1} - \lfloor x^{-1} \rfloor$) applied iteratively to ω or, equivalently, on the action of the shift map on the entries in the continued-fraction expansion of ω. In such studies the case of the golden-mean rotation number, for which $\omega = (\sqrt{5} - 1)/2$, often plays a pivotal role. This is perhaps not surprising given the simplicity of its continued fraction $[1, 1, 1, \ldots]$, with all entries equal to 1. For the golden-mean, renormalisation analysis frequently leads to fixed-point functional equations.

For example, renormalisation of correlations for the golden-mean Harper equation leads (see [46]) to the so-called strong-coupling fixed point, satisfying the functional equation:

$$f(z) = f(-\omega z)\, f(\omega^2 z + \omega) \tag{6.3}$$

where f is an analytic function with a pole of order 2 at $z = 1$ and $\omega = (\sqrt{5}-1)/2$. The construction of the strong-coupling fixed point involves first studying fixed points of the composition sum operator M in (6.1), namely:

$$Mf(z) = f(-\omega z) + f(\omega^2 z + \omega)$$

where f becomes now a branch of $\log(z-1)$ modulo an analytic function. (Here, of course, $\ell = 2$ and $a_1 = a_2 = 1$ in (6.2).)

Note that equation (6.3) is the fixed point case $f_n = f_{n-1} = f_{n-2} = f$ of the second-order multiplicative functional recurrence:

$$f_n(z) = f_{n-1}(-\omega z)\, f_{n-2}(\omega^2 z + \omega)\,. \tag{6.4}$$

Similarly, the associated linear recurrence:

$$f_n(z) = f_{n-1}(-\omega z) + f_{n-2}(\omega^2 z + \omega) \tag{6.5}$$

leads in turn to the operator M. The functional recurrences (6.4) and (6.5) arise in several contexts involving the golden-mean rotation number, in particular in connection with the Ketoja-Satija orchid flower for the generalised Harper equation [29, 45], and, with piecewise constant functions f_n, in the analysis of quantum two-level systems [14], barrier billiards [8] and strange non-chaotic attractors [16, 43]. We refer to the comprehensive book by Feudel *et al* [15] for a full discussion of the applications of renormalisation theory in strange non-chaotic attractors and to [48] for an overview of applications of equations (6.4) and (6.5).

For other quadratic irrationals with constant continued-fraction expansion, say $\omega = [a, a, a, a, \ldots]$ where the integer $a \geq 1$, we obtain the multiplicative and additive fixed-point equations

$$f(z) = \left(\prod_{k=0}^{a-1} f(-\omega z - k)\right) f(\omega^2 z + a\omega)\,, \quad f(z) = \left(\sum_{k=0}^{a-1} f(-\omega z - k)\right) + f(\omega^2 z + a\omega) \tag{6.6}$$

again with associated functional recurrences. See [44, 10] for applications in this case.

We now return to the golden-mean case. In [46], a rigorous analysis established *inter alia* the existence and properties of a unique solution of (6.3) under the constraints of that physical situation, and provided an explicit expansion for the strong-coupling fixed point. Indeed, writing $\phi_1(z) = -\omega z$ and $\phi_2(z) = \omega^2 z + \omega$, this fixed point of M has the form:

$$f(z) = \lambda \left[\log\left(\frac{1-z}{1-\omega}\right) + \sum_{k=1}^{\infty} \sum_{\substack{i_1 \ldots i_k \\ i_1=1}} \log \frac{1 - \phi_{i_1} \circ \ldots \circ \phi_{i_k}(z)}{1 - \phi_{i_1} \circ \ldots \circ \phi_{i_k}(\omega)} \right]$$

where $i_j \in \{1, 2\}$, $\lambda \in \mathbb{C}$, log is the principal branch of the logarithm, and $\circ$ signifies functional composition.

The proof in [46] is non-trivial and also depends on properties of the golden mean so that its generalisation is not obvious. The theory presented in this chapter provides a general framework for the construction of fixed-points of composition sum operators, which not only illuminates the results of [46], but enables the construction of fixed-points of other renormalisation operators in a simplified and unified manner.

We note that the maps ϕ_1, ϕ_2 form an Iterated Function System (see eg Falconer [13]) whose properties are well known. In particular this system possesses a (non-fractal attractor) in the complex plane which consists of the interval $I = [-\omega, 1]$ of the real line. I has the self-affine partition $I = (\phi_1 I) \bigcup (\phi_2 I)$ - see Figure 6.1.1.

The organisation of the rest of this chapter is as follows. We first describe in section 6.3.1 a general theory for the construction of fixed points of linear operators on vector spaces. In section 6.3.2 we introduce formally the composition sum operators (CSOs), describe their properties, and define the function spaces of analytic functions on which we shall work. We introduce the idea of a seed function, which we will use extensively in the construction of fixed points of these operators. In 6.4, we apply the theory to construct fixed points of CSOs in the constant coefficient affine case (which we simply call affine CSOs). Finally, in 6.5, we show how the theory can be applied to construct fixed-points arising in the renormalisation theory of quasiperiodic systems. This final section uses a construction method derived from the methods in [46].

6.3 Preliminaries

6.3.1 Fixed point theory on vector spaces

In this section we describe the formal abstract setting for our construction of fixed-points of linear operators on vector spaces. Our goal is to derive from a given linear operator T, a *fixed point operator* $\widehat{T}$ which maps its domain (called the *seed space* of T) to the *fixed point space* of T (denoted

$FP(T)$).

Although we have in mind applications to composition operators on spaces of complex analytic functions with various types of singularities, the theory is quite general and may used in cases in which linear operators act on a vector space that may be decomposed into a direct sum of subspaces, one with a well defined Banach space structure and no non-zero fixed points, and the other consisting of vectors lacking a finite norm, but which are prototype fixed points or "seeds". Although, set in this general context, the theory is straightforward, its power lies in its application to construct fixed points of renormalisation operators and other operators, in which the fundamental structure of the fixed points are evident, but the precise detail is not.

Let F be a vector space and let $G \subset F$ be a proper non-zero subspace of F. In many cases G is equipped with norm $\|\cdot\|$ which endows G with a Banach space structure, but this is not necessary for the general theory. Let $T : F \to F$ be a linear operator and let $\overline{T}$ denote the operator $I - T$ on F. We note that $f \in F$ is a fixed point of T if, and only if, it is in the kernel of $\overline{T}$.

We assume that T satisfies the following two properties.

P1 $\overline{T}(F) \subseteq G$, so that $\overline{T}$ maps the whole of F into the subspace G.

P2 The restricted operator $\overline{T}_{|G}$ is invertible on G so that $T^+ = \overline{T}_{|G}^{-1}$ exists and maps G to G.

We note three points. First, when G has a Banach space structure with norm $\|\cdot\|$, then the second condition is satisfied when $T_{|G}$ is a contraction with $\|T\| < 1$ (but this is not a necessary condition). Second, although these two conditions are very general, one can often think of G as the well-behaved non-singular part of F and $F \backslash G$ as being the singular or unbounded part of F, which provides seeds for the construction of non-zero fixed points of T. Third, writing F explicitly as the direct sum $F = G \oplus S$, we will see below that the vector space S is a subspace of the seed space which is mapped one to one to the fixed points of T by the fixed point operator.

The following is the principal result for the construction of fixed points of the operator T.

Theorem 6.3.1. *Let F be a vector space and $T : F \to F$ be a linear operator satisfying the conditions P1 and P2 above. Then the linear operator $\widehat{T} : F \to F$ given by*

$$\widehat{T} = I - T^+\overline{T}$$

maps F to the subspace $FP(T)$ of fixed points of T, and $\widehat{T}$ induces a vector space isomorphism from the factor space F/G to $FP(T)$. In the case when $F = G \oplus S$, $\widehat{T}$ induces an isomorphism from S to $FP(T)$.

The proof of this theorem is quite straightforward and belies the utility and power of the theorem itself.

Proof. Let $f \in F$ and consider $\widehat{T}f$. Then $\overline{T}(\widehat{T}f)) = \overline{T}f - \overline{T}(T^+\overline{T})f = \overline{T}f - \overline{T}f = 0$ so that $\widehat{T}f$ is a fixed point of T. Conversely, let $f \in F$ be a fixed point of T. Then $\overline{T}f = 0$ so that $\widehat{T}f = f$, and so

$f \in \widehat{T}F$. It is straightforward to show that $FP(T)$ is a linear subspace of F.

Now let us abuse notation slightly and also denote by $\widehat{T}$ the map $\widehat{T} : F/G \to FP(T)$ given by $\widehat{T}[f] = \widehat{T}f$, for $[f]$ an element of F/G. It is straightforward to show that $\widehat{T}$ is a linear map and we note that this map is well defined because G is in the kernel of $\widehat{T}$. It is immediate that Im $\widehat{T}$ $= FP(T)$. Now let $[f] \in F/G$ be in ker $\widehat{T}$. Then $\widehat{T}[f] = [0]$, so that $[f] = T^{+}\overline{T}[f]$ whence $f \in G$ and so $[f] = [0]$. It follows that $\widehat{T}$ is a vector space isomorphism.

Finally, in the case $F = G \oplus S$, S is isomorphic to F/G via the natural inclusion, and so $\widehat{T}$ induces a vector space isomorphism S to $FP(T)$. □

Note that $\widehat{T}$ is now our fixed point operator derived from T, and its seed space (domain) is $\overline{T}^{-1}G$[1], i.e. f is a seed if, and only if, $f \in \overline{T}^{-1}G$. This also means $\widehat{T}f = f + g$ for $g = -T^{+}\overline{T}f \in G$.

There is also a straightforward but important extension which allows us to extend the operator $\widehat{T}$.

Corollary 6.3.2. *Suppose that condition P2 holds, but not P1, so that we do not necessarily have* $\overline{T}(F) \subseteq G$. *Suppose, instead, that for some* $f \in F, \overline{T}f \notin G$, *but* $\overline{T}(T^k f) \in G$ *for some integer* $k \geq 1$. *Writing* $f_k = \sum_{i=0}^{k-1} \overline{T}(T^i f)$, *then* $\overline{T}(f - f_k) \in G$ *and* $\widehat{T}(f - f_k)$ *is a fixed point of* T. *Moreover, the fixed point is independent of the choice of* k.

Proof. Since $f_k = \sum_{i=0}^{k-1} \overline{T}(T^i f) = f - T^k f$, it is immediate that $\overline{T}(f - f_k) = \overline{T}(T^k f) \in G$, and so $f - f_k$ is a seed and we may now apply theorem 6.3.1. The final statement follows from the observation that, if $\tilde{k} \geq k$, then $\overline{T}T^{\tilde{k}} f = T^{\tilde{k}-k}\overline{T}T^k f \in G$, since $T(G) \subseteq G$. It follows that $\widehat{T}(f - f_{\tilde{k}}) - \widehat{T}(f - f_k) = \widehat{T}(f_k - f_{\tilde{k}}) = 0$, since $f_k - f_{\tilde{k}} \in G$. □

We call a vector f satisfying the hypotheses of corollary 6.3.2 a *generalised seed.*

As a simple application of theorem 6.3.1, we consider the operator T on real functions $c(x)$ defined on $[-1, 1]$

$$Tc(x) = c\left(\frac{x-1}{2}\right) - c\left(\frac{1-x}{2}\right).$$

This operator arises from the zero-shear base case of a the stretch-fold-shear toy model in kinematic dynamo theory, studied in detail by Gilbert [18, 19].

Now, for integer $n \geq 1$, let P^o_{2n-1} denote the real vector space of odd polynomials of degree at most $2n-1$. Then, evidently, $TP^o_{2n-1} \subseteq P^o_{2n-1}$, and, indeed, T has an upper-triangular matrix with respect to the standard basis $\{x, x^3, \ldots, x^{2n-1}\}$, from which the spectrum of T restricted to P^o_{2n-1} is readily obtained. Let us now consider the operator T from the viewpoint of theorem 6.3.1. Writing $P^o_{2n-1} = P^o_{2n-3} \oplus < x^{2n-1} >$ and $T_2 = 4^{n-1}T$, then it is straightforward to verify that the hypotheses of theorem 6.3.1 are satisfied with $G = P^o_{2n-3}$ and $S =< x^{2n-1} >$, from which the spectrum and

[1] Here $\overline{T}^{-1}G = \{f \in F : \overline{T}f \in G\}$, as $\overline{T}$ will generally not be invertible on the whole of F.

eigenfunctions of T on P^o_{2n-1} may be calculated. In fact, this is also the spectrum of T acting on a more general space of analytic functions on which T is compact. See [18, 19] for details.

6.3.2 Composition sum operators

We can now apply the general theory of the previous section to help us identify fixed points of the particular class of operators we call *composition sum operators*. The operators M and T introduced above are examples of this class.

In what follows we will adopt the convention of writing $f\alpha$ to denote the composition of f with α defined by $(f\alpha)(x) = f(\alpha(x))$. When we need to indicate a scalar multiplication that could be confused with a composition we will use the "dot" notation, $a.f$.

Definition 6.3.3. Composition Sum Operators

Let $\alpha : D \longrightarrow D$ be a map of a complex domain into itself, and let F be a ring of complex-valued functions defined on D. In practical applications, D is frequently a disc and F a space of functions analytic on a dense subset of D (ie admitting various types of singularities on D).

We define the operator α^* on F by $\alpha^* f = f\alpha$. We call α^* a *COMPOSITION OPERATOR* on F.

A *COMPOSITION SUM OPERATOR (CSO)* on F is an operator $T = \sum_{i=1}^{\ell} a_i\alpha_i^*$ on F where $a_i \in F$, $a_i \neq 0$, α_i^* is a composition operator on F, and $Tf = \left(\sum a_i\alpha_i^*\right) f = \sum a_i.f\alpha_i$. We call the positive integer ℓ the *length* of the CSO, and we assume that $\alpha_i \neq \alpha_j$, for $i \neq j$. When it is clear from the context, we suppress the explicit range $i = 1, \dots, \ell$. We call D a *BASE DOMAIN*, noting that by an abuse of notation we may also regard T as a formal sum operators on many base domains and function spaces.

We say that $T = \sum_{i=1}^{\ell} a_i\alpha_i^*$ is an *AFFINE CSO* if each a_i is a constant, and each α_i is an affine contraction, i.e. $\alpha_i(z) = s_i(z - z_i) + z_i$, where $z_i \in D$ and $|s_i| < 1$, $s_i \in \mathbb{C}$.

Example 6.3.4. The operator M of (6.1) is an affine CSO. To see this, note that $M = \phi_1^* + \phi_2^*$ where $\phi_1^*(z) = -\omega z$, $\phi_2^*(z) = \omega^2 z + \omega = \omega^2(z-1) + 1$. So M is a CSO of length 2, with constant coefficients $a_1 = 1, a_2 = 1$ and affine contractions with $s_1 = -\omega$, $s_2 = \omega^2$.

In the case when F is a Banach space with norm $||\cdot||$, we have $||Tf|| = ||\sum a_i.f\alpha_i|| \leq \sum ||a_i||\,||f||$ so $||T|| \leq \sum ||a_i|| < \infty$ so a CSO is also a bounded linear operator on these spaces. However, as we discussed in the introduction, the most interesting cases occur when T is an operator on a space which includes points without a finite norm (e.g. functions with singularities).

The fixed points of the constituent composition operators α_i of a CSO play an important role in the theory. We introduce the important notion of fixed point independence which characterises a class of CSOs of interest.

Definition 6.3.5. Fixed Point Independence

Let $T = \sum_{i=1}^{\ell} a_i \alpha_i^*$ be a composition operator on a space of functions defined on a domain D. We say α_i is FIXED POINT INDEPENDENT in T if $FP(\alpha_i) \bigcap \overline{\alpha_j D} = \emptyset$ for all $j \neq i$, ie every fixed point of α_i is separated from the image the base domain under each α_j. If α_i is fixed point independent in T for each i, then we say T is itself fixed point independent.

Example 6.3.6. The CSO M is fixed point independent on the base domain $D = (-\omega^{-1} + \epsilon, \omega^{-2} - \omega^{-1}\delta) + i\mathbb{R}$ for any $\delta \geq \epsilon > 0$ and $\delta + \omega\epsilon \leq \omega^{-2}$. To see this, first note that ϕ_1, ϕ_2 have fixed points 0, 1 respectively. The result follows by observing that $1 \notin \phi_1(D) = (-\omega^{-1} + \delta, 1 - \omega\epsilon) + i\mathbb{R}$ and $0 \notin \phi_2(D) = (\omega^2\epsilon, \omega^{-1} - \omega\delta) + i\mathbb{R}$.

In the next section we develop the theory of seed functions to construct fixed points of CSOs with Poles, Essential Singularities and Simple Logarithmic singularities (PESL singularities - see definition 6.4.2). We concentrate on these as they currently seem the most significant; however the techniques presented are readily extended to other types of singularity such as algebraic singularities.

6.3.3 Unbounded seed functions over Banach spaces of complex functions

In this section we look at conditions under which an unbounded function f can be a seed for a CSO T over a Banach function space G. Recall that a seed of T over G is an element f of the domain of T for which $\overline{T}f \in G$. This is a strong condition as we shall see. When T operates on functions, we will call its seeds *seed functions*.

We will show later that apart from a small set of CSOs which admit polynomials as fixed points, there are no non-trivial analytic fixed points of an affine CSO in G. All other non-zero fixed points therefore have singularities of some sort. For some CSOs with real coefficients the singularities can be discontinuities (see [43] for an example), but in the context of CSOs defined on spaces of complex analytic functions (with singularities) we are most interested in PESL singularities (see 6.4.2).

We start with a formal definition of unbounded points as these play an important role in the theory.

Definition 6.3.7. The UNBOUNDED SET of a function

Let $f : U \to \mathbb{C}$ for some open subset U of $\mathbb{C}$. We say that $z_0 \in U$ is an UNBOUNDED POINT of f if z_0 has no neighbourhood $N(z_0) \subseteq U$ on which f is bounded, i.e. for any neighbourhood $N(z_0)$ in U $\sup_{z \in N(z_0)} |f(z)| = \infty$. The unbounded set of f in U, denoted $\mathrm{unb}_U(f)$ is the set (possibly empty) of unbounded points of f in U. When U is clear we will simply write $\mathrm{unb}(f)$.

The significance of the unbounded set lies in the following:

Proposition 6.3.8. *If $f : U \to \mathbb{C}$ is a seed function for a CSO T over a Banach function space G, then* $\mathrm{unb}_U(f) = \mathrm{unb}_U(Tf)$, *Proof.* Recall that f is a seed function if $\overline{T}f \in G$. Hence $\overline{T}f = f - Tf$ is bounded. It follows that f, Tf must share their unbounded points. □

In principle, $\mathrm{unb}_U(f)$ may be large and composition operators may act and interact on the set in intricate ways which are beyond our current scope. We will restrict our attention to *simple* actions which we define as follows:

Definition 6.3.9. Functions which are SIMPLE under a CSO $T = \sum_{i=1}^{n} a_i\alpha_i$

Let $f : U \to \mathbb{C}$ for some open subset U of $\mathbb{C}$. We say f is simple under T on U if for any $z \in \mathrm{unb}_U(f)$:

1. For each $i \in \{1 \ldots n\}$, either $\alpha_i(z) = z$ or $\alpha_i(z) \notin \mathrm{unb}_U(f)$
2. If $\alpha_i(z) = z = \alpha_j(z)$ then $i = j$

Simple functions have an important independence property given by the following lemma:

Lemma 6.3.10. *Let $T = \sum_i a_i\alpha_i^*$ be a CSO, and let f be a seed over a Banach space G of functions. If f is simple under T, then $\mathrm{unb}(f) \subseteq \bigcup_i FP(\alpha_i)$, and each unbounded point of f is a fixed point of precisely one α_i.*

Proof. By proposition 6.3.8, if $z \in \mathrm{unb}(f)$ then $z \in \mathrm{unb}(Tf)$ so $Tf = \sum_i a_i.f\alpha_i$ is unbounded at z. Hence for at least one i, $f\alpha_i$ is unbounded at z, ie $\alpha_i z \in \mathrm{unb}(f)$. Since the unbounded set of f is simple, by definition $\alpha_i(z) = z$ and the i is unique. □

Note that if T is fixed point independent (see definition 6.3.5) then item (2) in definition 6.3.9 is satisfied for any f, and we need only check item (1) to determine if f is simple.

Example 6.3.11. Consider the affine CSO M. The fixed points of its composition operators are $\{0, 1\}$. So any simple seed or fixed point of M is unbounded on at most $\{0, 1\}$.

6.4 Fixed points of affine CSOs with PESL singularities

In this section we will study fixed points of affine CSOs acting on certain complex function spaces which we call PESL spaces (for reasons to be clarified shortly). We wish to apply our results about fixed points of linear operators to complex functions with unbounded points, but this raises a technical issue which we address first.

6.4.1 Technical preliminaries

In our study we will use tools of linear algebra together with those of complex analysis. In the linear algebra world our primary objects of study are functions defined over a common domain; in the complex analysis world our primary objects are holomorphic functions. The techniques of both worlds work well together when we are dealing with holomorphic functions over a common domain, and elsewhere this is very often the case. Unfortunately in our case, our functions may be non-holomorphic at any point of the domain, and we need to take some care in bridging between the two worlds. Fortunately once recognised, this is not a difficult thing to do, and there are a number of possible approaches. The approach we take here has the benefit of preserving the primary objects of study in both worlds. We will be quite formal in our approach, as our experience is that not doing so can lead to subtle misunderstandings.

Definition 6.4.1 (Holomorphic Extensibility)**.** Given a non-empty open subset $U \subseteq \mathbb{C}$:

Let $F^+(U)$ be the set of partial functions with complex values on U, ie for $f \in F^+(U)$ the domain of f is a subset of U (possibly empty or full). Let $H^+(U)$ be the subset of holomorphic functions in $F^+(U)$[2].

$F(U) \subset F^+(U)$ be the vector space of total functions (ie defined on the whole of U), and $H(U)$ the subspace of holomorphic functions on U.

For $f \in F^+(U)$ the HOLOMORPHIC SUPPORT U_f of f is the set of points in U on which f is holomorphic, and the HOLOMORPHIC PART of f is $f|_{U_f} \in H^+(U)$. The DERIVATIVE f' of f exists when $U_f \neq \emptyset$ and is defined to be $(f|_{U_f})' \in H^+(U)$. The SINGULARITY SET X_f of f in U is the set of points in U at which f is not holomorphic.

We say $f, g \in F^+(U)$ are HOLOMORPHICALLY EQUAL and write $f =_H g$ if their holomorphic parts are equal, ie $U_f = U_g$ and $f|_{U_g} = g|_{U_g}$.

Also we say f is a HOLOMORPHIC EXTENSION of g if $U_f \supseteq U_g \neq \emptyset$ and $f|_{U_g} = g|_{U_g}$, and g is HOLOMORPHICALLY EXTENSIBLE at $z \in U$ if there is a holomorphic extension f of g such that $z \in U_f$. If $z \in X_g$ we call z an EXTENSIBLE SINGULARITY of g. (Note that f itself is not required to be holomorphic on the whole of its domain).

Note that $F^+(U), H^+(U)$ are not vector spaces, because if the domain of f is a strict subset of U then $f - f \neq 0|_U$. However observe that $H^+(U) \cap F(U) = H(U)$. Note also that a holomorphic extension is a generalisation of analytic continuation, and a removable singularity is also an extensible singularity.

The definition provides a framework of terms for linking our vector spaces with sets of holomorphic function. Crucially $=_H$ is an equivalence relation and any holomorphic function $h \in H^+(U)$ can be identified with an equivalence class of functions f in the vector space $F(U)$ which have h as their holomorphic part.

As an example, when we now talk of $h(z) = 1/z$ being holomorphic on $\mathbb{C}$ apart from a singularity at 0, we will formally regard h as being in $H^+(\mathbb{C})$ with domain $\mathbb{C}\backslash\{0\}$, and identify h

[2] For completeness we include the empty function in this set, though we will never use this fact.

with the equivalence class of functions $[f]$ in $F(\mathbb{C})$ defined by $f(z) = 1/z$ for $z \neq 0$.

6.4.2 PESL spaces

We are now able to construct very general vector spaces of unbounded functions. We will proceed to construct examples of spaces which contain functions with SINGULARITIES OF PESL TYPE (*P*oles, *E*ssential singularities, and *S*imple *L*ogarithmic singularities - the latter are defined below). Due to the possible presence of many overlapping branch cuts, these spaces also contains functions whose holomorphic support can have complicated topology, and in turn we find this also requires us to allow for the inclusion of functions with extensible singularities. The particular spaces we construct here are in fact slightly larger than minimal, but admit an elegant definition.

Definition 6.4.2 (PESL space). Given an open connected set U, let $f \in F(U)$ be a function whose holomorphic support U_f is dense in U. We say f is a PESL function if its derivative $f' \in H^+(U)$ has a holomorphic extension $f^* \in F(U)$ which has at most finitely many (hence isolated) singularities in U. We call the set $P(U) \subset F(U)$ of PESL functions on U the PESL space on U.

Proposition 6.4.3. *The PESL functions $P(U)$ on U form a vector subspace of $F(U)$*

Proof. Clearly $0 \in P(U)$and if $f \in P(U)$ then $\lambda f \in P(U)$. It remains to show that if $f, g \in P(U)$ the $f + g \in P(U)$. First, the holomorphic support U_{f+g} of $f + g$ contains $U_f \cap U_g$ which is dense-open in U, and so U_{f+g} is dense in U. But U_{f+g} is open by definition, and so is dense-open. Now let f^*, g^*be holomorphic extensions of f', g' with X_f, X_g finite. Then $X_{f^*+g^*} \subseteq X_f \cup X_g$ is also finite, and $f^* + g^*$ is holomorphic at any point that $f' + g'$is holomorphic so that $f^* + g^*$ is a holomorphic extension of $f' + g'$. Hence $f + g \in P(U)$. □

Note that we are here using complex analytic tools to define essential properties of $f \in F(U)$ and then using tools of linear algebra to define the PESL space. The following result shows that the PESL space is in fact just the PESL functions. That the PESL space include all functions with singularities of PESL type becomes clear from the following:

Definition 6.4.4. Let $f \in F(U)$ be a PESL function, z_0 a singularity of f^*, and let $P(z) = \sum_{j=1}^{\infty} a_{-j}(z - z_0)^{-j}$ be the principal part of the Laurent series of f^* around z_0. Then we classify the type of f at z_0 as follows:

1. If $P = 0$, and then z_0 is a removable singularity and f is holomorphically extensible at z_0 to another PESL function. We say f is of EXTENSIBLE type at z_0. Otherwise:
2. If f^* is a simple pole, ie $a_{-1} \neq 0$ and $P = a_{-1}(z-z_0)^{-1}$ then we say f is of SIMPLE LOGARITHMIC (SL) type at z_0 and that z_0 is a SIMPLE LOGARITHMIC SINGULARITY.

3. If f^* has a residue of 0 at z_0, ie $a_{-1} = 0$, then f^* still has a pole or essential singularity at z_0 and and we say f at z_0 is of POLE (P) or ESSENTIAL SINGULARITY (E) type respectively.

4. Otherwise we say f is of SUPERIMPOSED type at z_0, since $f^* = f_1^* + f_2^*$ where f_1 is of SL type and f_2 of P or E type at z_0.

We are now ready to apply the theory developed in previous sections. We will first establish the result previously claimed, namely that the only non-trivial analytic fixed points of an affine CSO are polynomials. We then proceed to consider fixed points in PESL space.

Recall that an affine CSO $T = \sum_{i=1}^{\ell} a_i \alpha_i^*$ is a CSO for which $a_1, a_2, \ldots, a_\ell$ are non-zero constants and the maps $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are affine contractions $\alpha_i(z) = z_i + s_i(z - z_i)$ on the complex plane, where the fixed points z_i and contraction rates s_i are all complex constants, and with $0 \le |s_i| < 1$. In many applications the z_i and the s_i are real, but the theory may be just as easily developed for complex z_i and s_i. With affine CSOs we are able to obtain a good theory for the construction of simple fixed points, drawing on the work of the previous sections.

We shall work in a fixed disc in the complex plane. Let $D = D_r$, the open disc of radius r about 0 in $\mathbb{C}$, and let $G(D_r)$ be the complex Banach space of functions g analytic on D_r with finite supremum norm $||g||_{\infty,r} = \sup\{|g(z)| \ : \ z \in D_r\}$. Let $R > 0$ be chosen so that for some $\delta > 0$, $\overline{\alpha_i(D_{R+\delta})} \subseteq D_{R-\delta}$ for all $i = 1, \ldots, \ell$. Because the α_i are contractions, this condition holds provided we take R sufficiently large. We write $D = D_R$, and $G = G(D_R)$ and $||g||_\infty = ||g||_{\infty,R}$, for $g \in G$.

6.4.3 Bounded analytic fixed points

Let us consider the affine composition sum operator $T = \sum_{i=1}^{\ell} a_i \alpha_i^*$. It is straightforward to verify that T is a linear operator on the complex Banach Space G. Moreover, for $m \ge 0$, we may differentiate m times the function Tg:

$$(Tg)^{(m)}(z) = \sum_{i=1}^{\ell} a_i s_i^m g^{(m)}(\alpha_i(z)) \,. \tag{6.7}$$

We now define an induced operator on G, which we denote by $T^{(m)}$, given by

$$T^{(m)}\tilde{g}(z) = \sum_{i=1}^{\ell} a_i s_i^m \tilde{g}(\alpha_i(z)), \tag{6.8}$$

for $\tilde{g} \in G$. We have that

$$||T^{(m)}\tilde{g}||_\infty \le \sum_{i=1}^{\ell} |a_i||s_i|^m ||\tilde{g}||_\infty, \tag{6.9}$$

so that the operator norm $||T^{(m)}|| \le \sum_{i=1}^{\ell} |a_i||s_i|^m$. An immediate consequence is that there exists $m \ge 0$ such that $||T^{(m)}|| < 1$, a contraction. Finally, using Cauchy estimates, we see that if $g \in G$, then $g^{(m)}\alpha_i$ is also in G and, moreover, $||g^{(m)}\alpha_i||_\infty \le K||g||_\infty$, where $K = Rm!\,\delta^{-(m+1)}$, for $i = 1, \ldots, \ell$.

From these results, we may readily show that all non-trivial fixed points of T in G are polynomials. The proof is rather elegant. Indeed, suppose $g \in G$ is a fixed point of T, with $g \ne 0$. Then, for some $m \ge 0$, $0 \le ||g^{(m)}||_\infty = ||T^{(m)}g^{(m)}||_\infty \le ||T^{(m)}||\,||g^{(m)}||_\infty < ||g^{(m)}||_\infty$, a contradiction. It follows that g is zero or a polynomial of degree at most $m-1$.

Whether or not T has a polynomial fixed point depends on the precise values of the a_i and α_i. Indeed, for a polynomial $p(x) = p_0 + p_1x + \cdots + p_mx^m$, $p_m \ne 0$, it is clear that Tp is a polynomial of degree at most m. Inspecting the coefficient of x^m in $Tp(x) = p(x)$, we have

$$a_1s_1^m + a_2s_2^m + \cdots + a_\ell s_\ell^m = 1 \tag{6.10}$$

which is clearly a necessary condition for a polynomial fixed point of degree m. Conversely, suppose that (6.10) holds. Then if $p(x)$ is of degree m, $\overline{T}p$ is of degree at most $m-1$, and so $\overline{T}$ is degenerate and has non-trivial kernel. If q is in the kernel, then $Tq = q$. Hence T has a non-trivial space of polynomial fixed points if, and only if, (6.10) holds for one or more $m \ge 0$. Note that there is some $N > 0$ such that the condition does not hold for any $m \ge N$, and so the space of polynomial fixed points of T is of bounded maximum degree.

6.4.4 Fixed points with PESL singularities

We now assume that there are no polynomial fixed points, ie that

$$a_1s_1^j + a_2s_2^j + \cdots + a_\ell s_\ell^j \ne 1, \quad \text{for all } j \ge 0. \tag{6.11}$$

It is now evident that the only non-zero fixed points are necessarily singular on D. In what follows we restrict ourselves to considering functions with PESl singularities.

Let us consider first simple seeds f with unbounded isolated singularities. Since f is simple, every point of $\mathrm{unb}_D\, f$ is a fixed point of a unique α_i. Without loss of generality, we let $i = 1$ and we suppose that f has an isolated singularity at z_1, so that $f_{|D-\{z_1\}}$ is analytic on some neighbourhood of z_1. Let C_ϵ be the oriented circle $z_1 + \epsilon e^{i\theta}$ of radius $\epsilon > 0$ about z_1. Then if α_i is fixed point independent on D, and for ϵ sufficiently small, f is analytic inside and on $\alpha_i(C_\epsilon)$ for $i = 2, \ldots, \ell$.

Using the fact that $f - Tf$ is analytic and integrating along C_ϵ, we have, for integer $k \geq 0$,

$$0 = \int_{C_\epsilon} (z - z_1)^k (f(z) - Tf(z))dz = \int_{C_\epsilon} (z - z_1)^k (f(z) - a_1 f(\alpha_1(z)))dz \tag{6.12}$$

$$= \left(\int_{C_\epsilon} (z - z_1)^k f(z)dz - \int_{C_\epsilon} (z - z_1)^k a_1 f(\alpha_1(z))dz \right) \tag{6.13}$$

$$= \left(\int_{C_\epsilon} (z - z_1)^k f(z)dz - \int_{\alpha_1^{-1} C_\epsilon} s_1^{-(k+1)} (w - z_1)^k a_1 f(w)dw \right) \tag{6.14}$$

$$= 2\pi i f_{-(k+1)} \left(1 - s_1^{-(k+1)} a_1\right) . \tag{6.15}$$

In this calculation we have used Cauchy's integral theorem, together with a change of variable $w = \alpha_1(z)$. We have denoted by $f_{-(k+1)}$ the $(k + 1)$–th coefficient in the Laurent expansion of f about z_1.

We conclude for $k \geq 0$ that either $f_{-(k+1)} = 0$ or $a_1 = s_1^{(k+1)}$. If the latter condition holds, then we may have $f_{-(k+1)} \neq 0$ and $f(z) = (z - z_1)^{-(k+1)}$ is a seed function, from which a fixed point of T may be constructed, provided (6.11) holds. The construction is omitted here as it is similar to that given below for the logarithmic case.

The result also shows that essential singularities do not lead to fixed points of affine CSOs. For $\left(1 - s_1^{-(k+1)} a_1\right) = 0$ cannot hold for more than one $k \geq 0$, ruling out a non-finite principal part of f at z_1.

We now give the construction of a fixed point of T in the case when the seed function f is of simple logarithmic type at z_i (see definition 6.4.2) where $z_i \in D$, and where i is one of $1, 2, \ldots, \ell$. Again, without loss of generality, we take $i = 1$.

Our first observation is that we may take $f(z) = \log(z - z_1)$, since if $\tilde{f}(z) = \log(z - z_1) + g(z)$, where $g \in G$, then $\widehat{T} f = \widehat{T} \tilde{f}$. (Any convenient branch of the logarithm may be taken, although, to be specific, we choose the principal branch.) Again we let α_1 be fixed point independent on D so that, for $j \neq 1$, $z_1 \notin \overline{\alpha_j(D)}$. Therefore $f(z) - Tf(z) = \log(z - z_1) - a_1 \log(\alpha_1(z) - z_1) + g(z)$, where $g \in G$, and, since $\log(z - z_1) - a_1 \log(\alpha_1(z) - z_1) = \log(z - z_1) - a_1 \log(z_1 + s_1(z - z_1) - z_1)$ $= (1 - a_1) \log(z - z_1) - a_1 \log s_1$, it follows that $f - Tf \in G$ if, and only if, $a_1 = 1$.

Let us now assume that $a_1 = 1$ and $f(z) = \log(z - z_1)$. For convenience we consider separately the cases when T is a contraction on G and when T is not a contraction on G.

The first case is easily handled directly by appealing to theorem 6.3.1. Let $F = \langle \log(z - z_1) \rangle \oplus G$. Then $T : F \to F$ satisfies the hypothesis of theorem 6.3.1, from which we conclude immediately that $\hat{T} F$ is the space of fixed-points of T in F.

The second case may be handled by differentiating the operator T, say m times, until it is a contraction, appealing to theorem 6.3.1 for the induced operator $T^{(m)}$, and then integrating up to obtain a fixed point of T. Specifically, let $m \geq 1$ be such that $\sum_{i=1}^{\ell} |a_i||s_i|^m < K < 1$ and let

$f_m(z) = (m-1)!(-1)^{m-1}(z-z_1)^{-m}$. Then

$$T^{(m)} f_m(z) - f_m(z) = \sum_{i=1}^{\ell} a_i s_i^m f_m(\alpha_i(z)) \; - f_m(z) = \sum_{i=2}^{\ell} a_i s_i^m f_m(\alpha_i(z)), \tag{6.16}$$

as may readily be ascertained by direct calculation. The right-hand side is in G (since $z_1 \notin \overline{\alpha_i(D)}$ for $i = 2, \ldots, \ell$), so that f_m is a seed function for $T^{(m)}$. Moreover, $I - T^{(m)}$ is invertible in G because $||T^{(m)}|| \leq K < 1$. We may therefore apply theorem 6.3.1 with $F = < f_m > \oplus G$ to obtain a one-dimensional subspace of fixed points $< \widehat{f_m} >$ of $T^{(m)}$ in F.

To obtain a fixed point of T, we integrate m times, although we must then handle a polynomial of degree at most $m-1$ that arises from the constants of integration. Specifically, let us define the integration operator $I : G \to G$ by the integral on the line segment $[0, z]$ for $z \in D$:

$$I(g)(z) = \int_0^z g(w)dw\,. \tag{6.17}$$

Denoting the m–th iterate of I by I^m, and noting that $\widehat{f_m} - f_m \in G$, we may define the function $f + I^m(\widehat{f_m} - f_m)$ which we denote $\hat{f}$. The function $\hat{f}$ is not necessarily a fixed point of T. However, differentiating $T\hat{f} - \hat{f}$ m times, we obtain

$$\left(T\hat{f} - \hat{f}\right)^{(m)} = \left(Tf - f + TI^m(\widehat{f_m} - f_m) - I^m(\widehat{f_m} - f_m)\right)^{(m)} \tag{6.18}$$

$$= (T^{(m)} f_m - f_m) + T^{(m)}(\widehat{f_m} - f_m) - (\widehat{f_m} - f_m) \tag{6.19}$$

$$= 0, \tag{6.20}$$

since $T^{(m)}\widehat{f_m} = \widehat{f_m}$. It follows that $T\hat{f} - \hat{f} = q_m$, where q_m is a polynomial of degree at most $m-1$. Now let $p_m = (I-T)^{-1} q_m$, a polynomial of degree at most $m-1$, the inverse existing because of (6.11). Then we have immediately that $T(\hat{f} + p_m) - (\hat{f} + p_m) = 0$, so that $\hat{f} + p_m$ is a fixed point of T.

6.4.5 Summary

We have proved the following result:

Theorem 6.4.5. *Let T be an affine composition sum operator given by*

$$T = \sum_{i=1}^{\ell} a_i \alpha_i^*,$$

where $\ell \geq 2$ is an integer, and for $i = 1, \ldots, \ell$, $a_i \in \mathbb{C}$, and $\alpha_i(z) = s_i(z - z_i) + z_i$ are affine contractions. Let $R > 0$ be such that there exists $\delta > 0$ with $\overline{\alpha_i(D_{R+\delta})} \subseteq D_{R-\delta}$ for $i = 1, \ldots, \ell$.

Then

1. *T has a fixed point which is a non-zero polynomial if and only if $a_1 s_1^m + a_2 s_2^m + \cdots + a_\ell s_\ell^m = 1$ for some integer $m \geq 0$. If there are polynomial fixed points, there is also a maximum integer m satisfying the constraint, and all the fixed points are then of degree at most m.*

2. *If there are no polynomial fixed points[3], but for some $1 \leq i \leq l$, α_i is fixed point independent on D_R (ie $z_i \notin \bigcup_{j \neq i} \overline{\alpha_j(D_R)}$), then:*

   (a) If $a_i = s_i^k$, for some $k \geq 1$, then T has a fixed point f of the form

   $$f(z) = (z - z_i)^{-k} + g(z)$$

   where $g \in G$ is analytic and bounded in D_R, and also uniquely determined.

   (b) If $a_i = 1$, then T has a fixed point f of the form

   $$f(z) = \log(z - z_i) + g(z)$$

   where $g \in G$ is analytic and bounded in D_R, and also uniquely determined.

Moreover if T is fixed point independent on D_R, then every simple fixed point of T whose singularities are of PESL type is necessarily a linear combination of fixed points satisfying the conditions above.

Example 6.4.6. We consider the affine CSO M. The requirement of the theorem on R is satisfied by putting $R = \omega^{-1} - \delta$ for some small δ, and then by 6.3.6 it is also the case that M is fixed point independent on D_R (ie both its composition operators are fixed point independent on D_R). Also from 6.3.6, the CSO M has length 2 with $a_1 = a_2 = 1$ and $s_1 = -\omega$, $s_2 = \omega^2$. By the theorem, it has no polynomial fixed points, and its simple fixed points with PESL singularities have the form $\lambda \log z + \mu \log(z - 1) + g(z)$ with g analytic and bounded in D_R and uniquely determined. In the next section we will determine g.

6.5 An alternative (elementary) approach

In the previous section we developed some general theory on the existence and nature of fixed points of affine CSOs. The approach also provides a method for constructing these fixed points concretely. However it requires the use of complex calculus. We will present here an alternative approach to construction which uses only elementary methods. We will use the operator M from (6.1) as a concrete example for the construction.

[3]This condition guarantees the invertibility of $I - T$, and hence the existence of PESL fixed points. However if polynomial fixed points do exist, the possibility of PESL fixed points is not ruled out, and if they do exist they will satisfy the conditions given above for a_i.

In the previous approach we derived a contraction operator from the initial CSO by differentiating a number of times. In this approach we also derive a contraction, but this time we obtain it by a simple algebraic operation. This is more in the spirit of the work in [46, 10]. We start by developing a general theory of CSOs acting on ℓ_1 spaces of holomorphic functions. (This is complementary to the theory of the previous section).

6.5.1 CSOs acting on ℓ_1 spaces of holomorphic functions

For $R > 0$, let G_R denote the complex Banach space of holomorphic functions on the open disc $D_R = \{z : |z| < R\}$ with finite ℓ_1-norm $\left\|\sum_{n=0}^{\infty} c_n z^n\right\|_R = \sum_{n=0}^{\infty} |c_n| R^n$. For $n \geq 0$, we denote by Z_n the basis function $Z_n : z \longmapsto z^n$, which has norm R^n. The set $\{Z_n : n = 0, 1, 2, \dots\}$ forms a basis for G_R. We note the following standard lemma, which we include for completeness.

Lemma 6.5.1. *Let T be a bounded linear operator on the Banach space G_R of holomorphic functions and let $K > 0$. Then the induced operator norm $\|T\|_R \leq K$ if, and only if, $\|TZ_k\|_R \leq K\,\|Z_k\|_R$ for all $k \geq 0$. The result also holds when the inequality is replaced with a strict inequality.*

It follows that T is a contraction on G_R with contraction rate $K < 1$ if, and only if, it contracts each basis function Z_k with contraction rate K.

Proof. First suppose $\|T\|_R \leq K$. Since $Z_k \in G_R$, $\|TZ_k\|_R \leq \|T\|_R \|Z_k\|_R \leq KR^k$, as required. We now prove the converse. Let $f = \sum_{r=0}^{\infty} a_r Z_r \in G_R$. Since T is bounded, hence continuous, $Tf = T\lim_{n\to\infty} \sum_{r=0}^{n} a_r Z_r = \lim_{n\to\infty} T\sum_{r=0}^{n} a_r Z_r = \lim_{n\to\infty} \sum_{r=0}^{n} a_r TZ_r$ and so it follows that $\|Tf\|_R = \lim_{n\to\infty} \|\sum_{r=0}^{n} a_r TZ_r\|_R \leq \lim_{k\to\infty} \sum_{r=0}^{k} |a_r| KR^r = K\|f\|_R$ whence $\|T\|_R \leq K$, as claimed. This completes the proof. □

One particular feature of affine CSOs is that they are contractions on the basis functions Z_k for k sufficiently large, as is shown by the following result.

Lemma 6.5.2. *Let $T = \sum_{i=1}^{\ell} a_i \alpha_i^*$ be a CSO with a_i constant (and non-zero), $\alpha_i(z) = s_i z + t_i$ where $|s_i| < 1$. Let $s = \max_i\{|s_i|\}$, and let $\mu \in \mathbb{R}$ satisfy $s < \mu \leq 1$.*

Then there exists $R_0 \geq 0$, and integer $N > 1$ such that

1. *$\|TZ_n\|_R < \mu^n R^n$ for all $R > R_0$ and all $n \geq N$.*

2. *For $0 \leq n < N$, $\|TZ_n\|_R < \mu R^n$ for all $R > R_0$, whenever $\left|\sum_{i=1}^{\ell} a_i s_i^n\right| < \mu$.*

Proof. We have $(TZ_n)(z) = \sum_i a_i (s_i z + t_i)^n = \sum_{r=0}^{n} z^r \sum_i a_i \binom{n}{r} s_i^r t_i^{n-r}$, hence it follows that

$$\|TZ_n\|_R \leq \sum_{r=0}^{n} \left|\sum_i a_i \binom{n}{r} s_i^r t_i^{n-r}\right| R^r = R^n \sum_{r=0}^{n} \binom{n}{r} \left|\sum_i a_i s_i^r \left(\frac{t_i}{R}\right)^{n-r}\right| \tag{6.21}$$

Therefore

$$\|TZ_n\|_R \le R^n \sum_i |a_i| \left(|s_i| + \left|\frac{t_i}{R}\right|\right)^n \tag{6.22}$$

from which we see that if $|s_i| + \left|\frac{t_i}{R}\right| < \mu \le 1$ for all i, then we can find $N \ge 0$ so that $\|TZ_n\| < \mu^n R^n$ for $n \ge N$. The condition on R equates to $R_0 = \max_i\{\frac{|t_i|}{\mu - |s_i|}\}$.

We now consider $n < N$ for $N > 1$. From the above:

$$\|TZ_n\|_R \le R^n \left(\left|\sum_i a_i s_i^n\right| + \sum_{r=0}^{n-1} \binom{n}{r} \left|\sum_i a_i s_i^r \left(\frac{t_i}{R}\right)^{n-r}\right| \right) \tag{6.23}$$

Since n is now bounded, for any $\epsilon > 0$, we can choose R large enough to give $\|TZ_n\|_R \le R^n \left(\left|\sum_i a_i s_i^n\right| + \epsilon\right)$.

Hence if $\left|\sum_i a_i s_i^n\right| < \mu \le 1$ for $0 \le n < N$, we will have $\|TZ_n\|_R < \mu R^n$ for large enough R. □

The following corollaries are immediate.

Corollary 6.5.3. *If T is an affine CSO with each $|s_i| < 1$ and, for each $n \ge 0$, $\left|\sum_i a_i s_i^n\right| < 1$, then T is a contraction on G_R for large enough R.*

Corollary 6.5.4. *If T is an affine CSO with each $|s_i| < 1$ and, for each $n \ge 1$, $\left|\sum_i a_i s_i^n\right| < 1$, then T_c is a contraction on G_R for large enough R, where c is a constant and $T_c f = Tf - c^*(Tf) = \sum_{i=1}^{\ell} a_i f \alpha_i - \sum_{i=1}^{\ell} a_i f \alpha_i(c)$.*

Proof. Note that for any constants c and a, ac^* is a degenerate affine CSO with $s = 0$, so that for $n \ge 1$ each sum $\left|\sum_i a_i s_i^n\right|$ is unchanged between T and T_c. But the sum for T_c is precisely 0 for $n = 0$, and so the previous corollary can be applied to T_c. □

For a seed function f, it may happen that the function $g = Tf - f \in G_R$ only for R in a restricted range. In these circumstances it may not be possible to apply Corollaries 6.5.3 and 6.5.4 directly. Instead we may have to iterate T several times so that the domain on which g is defined is extended to include D_R for R sufficiently large for Corollaries 6.5.3 and 6.5.4 to apply. That it is possible to do this follows from the fact that the α_i contract the whole of $\mathbb{C}$ uniformly.

Let us first note that there exists $R_0 \ge 0$ such that $\alpha_i(D_R) \subseteq (D_R)$ for each $i = 1, \ldots, \ell$ and each $R \ge R_0$. The following domain expansion lemma is straightforward to prove.

Lemma 6.5.5. *Let $g \in G_{R_1}$ for some $R_1 > R_0$. Then for each $R \ge R_1$ there exists an integer $K \ge 0$ such that $T^k g \in G_R$ for all $k \ge K$.*

Proof. The proof is a straightforward consequence of the fact that the α_i contract uniformly. If $g \in G_R$, then the result holds with $k = 0$. Otherwise, let $k \ge 1$ and consider a composition of k

contractions chosen from the α_i, possibly with repetition. The resulting composition $\alpha_{i_1}\dots\alpha_{i_k}$ is an affine map so we may write $\alpha_{i_1}\dots\alpha_{i_k}(z) = s_{i_1\dots i_k}z + t_{i_1\dots i_k}$, for $i_1,\dots,i_k \in \{1,\dots,\ell\}$. Now $t_{i_1\dots i_k} = \alpha_{i_1}\dots\alpha_{i_k}(0) \in D_{R_0}$. Moreover, since the α_i contract uniformly on $\mathbb{C}$, the sequence $s_{i_1\dots i_k} \to 0$ uniformly in k as $k \to \infty$. It follows immediately, that there must exist $k \geq 1$ such that $\alpha_{i_1}\dots\alpha_{i_k}(D_R) \subseteq D_{R_1}$ for all $i_1,\dots,i_k \in \{1,\dots,\ell\}$. It follows that for all k large enough, we have $T^k g \in G_R$, as claimed. □

In the application we shall consider in the next subsection, the operator T fails to be a contraction, because it is not a contraction on constant functions. To solve this problem, for any affine CSO T, we introduce a new derived operator which is a contraction, and which shares certain fixed points with T. We give the construction for general $\ell \geq 2$, although in our application we shall specialise to the binary case $\ell = 2$.

For $1 \leq j \leq \ell$, we define the operator T_j by

$$T_j f = Tf - \frac{1}{L}\sum_{i=1,i\neq j}^{\ell} a_i Tf(\alpha_i(z_j)), \quad L = \sum_{i=1,i\neq j}^{\ell} a_i \tag{6.24}$$

provided $L \neq 0$. We note that, if $a_j = 1$, a fixed point of T_j is also a fixed point of T. For, suppose $T_j f = f$. Then

$$f(z) = Tf(z) - \frac{1}{L}\sum_{i=1,i\neq j}^{\ell} a_i Tf(\alpha_i(z_j)). \tag{6.25}$$

Taking a weighted sum of this equation evaluated at $\alpha_i(z_j)$, gives

$$\sum_{i=1,i\neq j}^{\ell} a_i f(\alpha_i(z_j)) = \sum_{i=1,i\neq j}^{\ell} a_i\left(Tf(\alpha_i(z_j)) - \frac{1}{L}\sum_{k=1,k\neq j}^{\ell} a_k Tf(\alpha_k(z_j))\right) = 0. \tag{6.26}$$

Hence, using $\alpha_j(z_j) = z_j$ we obtain

$$\begin{aligned} f(z_j) &= Tf(z_j) - \frac{1}{L}\sum_{i=1,i\neq j}^{\ell} a_i Tf(\alpha_i(z_j)) \\ &= a_j f(z_j) + \sum_{i=1,i\neq j}^{\ell} a_i f(\alpha_i(z_j)) - \frac{1}{L}\sum_{i=1,i\neq j}^{\ell} a_i Tf(\alpha_i(z_j)), \end{aligned} \tag{6.27}$$

whence, since $a_j = 1$,

$$\frac{1}{L}\sum_{i=1,i\neq j}^{\ell} a_i Tf(\alpha_i(z_j)) = \sum_{i=1,i\neq j}^{\ell} a_i f(\alpha_i(z_j)) = 0, \tag{6.28}$$

from (6.26). It follows that $Tf = f$, as claimed.

If T is a binary CSO, then we can write the operator T_j as T_c where the modified operator T_c is given by $T_c f = Tf - Tf(c)$ with $c = \alpha_i(z_j)$ and where now $\{i, j\}$ is precisely $\{1, 2\}$. It is immediate that, for a binary CSO, any fixed point f of T_c is a fixed point of T and $f(c) = 0$ by (6.26).

6.5.2 Fixed points of the operator M

We now apply the theory we have developed to find fixed points of the operator M introduced in section 6.2. Recall that $M = \phi_1^* + \phi_2^*$ with $\phi_1(z) = -\omega z, \phi_2(z) = \omega^2 z + \omega$, where $\omega = \frac{1}{2}(\sqrt{5} - 1)$. For consistency with [46, 10], we use the notation $\phi_i = \alpha_i$ for $i = 1, 2$.

Now, in the notation used above, $s = \max(\omega, \omega^2) = \omega$. Let $\omega < \mu < 1$. It follows that, for N sufficiently large, $M(Z_n) \leq \mu^n R^n$, for $n \geq N$ and $R \geq R_0$. In fact it is readily seen that $N = 2$ suffices when $R \geq 1.9009$. For $0 \leq n \leq 1$, we calculate as follows. If $n = 1$, $\left|\sum_i a_i s_i^n\right| = \left|(-\omega) + \omega^2\right| = \omega^3 < 1$ and we have a contraction for R sufficiently large. It is straightforward to verify that is sufficient to take $R \geq 1.619$. We therefore need to choose $R \geq 1.9009$. However, for $n = 0$, $\left|\sum_i a_i s_i^n\right| = 2 > 1$ and we do not have a contraction.

Let us consider the operator M_c introduced at the end of the last section and given by $M_c f = Mf - Mf(c)$. Since $a_1 = a_2 = 1$, in this case, we can take in turn $j = 1, 2$ and choose, in turn, $c = c_1, c_2$, where $c_1 = \phi_1(1) = -\omega$ and $c_2 = \phi_2(0) = \omega$. We note that, for any c, M_c is itself a (degenerate) CSO. Indeed, $M_c = \phi_1^* + \phi_2^* - (\phi_1 c)^* - (\phi_2 c)^* = \sum_1^4 a_i \alpha_i$. The last two terms ($i = 3, 4$) are constants, so are degenerate affine contractions.

Let us now construct fixed points of M_c and hence of M. First we note that $M_c Z_0 = 0$, and, from the above calculations, we see that M_c contracts the functions $Z_n : z \longmapsto z^n$ for $n \geq 1$. We deduce from Corollary 6.5.4 that M_c is a contraction on G_R for large enough R. Hence M_c has no non-trivial holomorphic fixed points in G_R, for R large enough.

Our first task is to consider a space of seed functions for M. Since $a_1 = a_2 = 1$, we can look for seeds which are simple logarithmic at 0 or 1 (the fixed points of ϕ_1 and ϕ_2 respectively). Indeed for simple unbounded singularities, the space of seed functions for M_c is the span $< \log z, \log(z-1) >$. Hence, to find the fixed points of M_c we apply the previous theory to obtain a fixed point for each of the two basis seed functions $\log z$ and $\log(z - 1)$.

We calculate M acting on the two basis (generalised) seed functions. We have $M \log z = \log(-\omega z) + \log(\omega^2 z + \omega) = \log z + \log(1 + \omega z) + b_1$, where b_1 is a constant. Similarly $M \log(z-1) = \log(-\omega z - 1) + \log(\omega^2 z + \omega - 1)) = \log(z - 1) + \log(1 + \omega z) + b_2$, b_2 constant (where we have used $1 - \omega = \omega^2$). It follows readily that in both cases we have (using the notation of lemma 6.5.5) $\overline{M_c} \log z, \overline{M_c} \log(z - 1) \in G_{R_1}$, provided $1 = R_0 < R_1 < \omega^{-1}$. Because of the restriction on R_1, we cannot necessarily use theorem 6.3.1 directly, but if not we can use Corollary 6.3.2 instead. For convenience, in what follows, we write $f(z)$ to represent one of the generalised seed functions $\log z$ and $\log(z - 1)$ and we write $\tilde{f}(z) = \log(1 + \omega z)$.

Using Corollary 6.5.4 we choose R sufficiently large so that M_c is a contraction on G_R. If we can choose $1 < R < \omega^{-1}$, we can use theorem 6.3.1 directly to obtain a fixed point of M_c. Otherwise using lemma 6.5.5, we set $k \geq 0$ such that $M_c^k \overline{M_c} f \in G_R$. We note that in fact $k \geq 1$ since $\tilde{f} \notin G_R$, because $R \geq \omega^{-1}$. It also follows that $M_c^k(f)$ is a seed function since $\overline{M_c} M_c^k f \in G_R$. A fixed point of M_c is now obtained from this seed function by applying Corollary 6.3.2.

We can obtain some explicit expansions for the fixed points. From the above calculations, we have $Mf = f + \tilde{f} + b$, where b is a constant, whence $M_c f = I_c f + I_c \tilde{f}$. Hence we readily obtain for $k \geq 1$ that $M_c^k f = I_c f + I_c \tilde{f} + \sum_{n=1}^{k-1} M_c^n \tilde{f}$, and so $\overline{M_c} M_c^k f = -M_c^k \tilde{f}$. We conclude that $M_c^k \tilde{f} \in G_R$.

Let

$$f_* = I_c f + I_c \tilde{f} + \sum_{n=1}^{\infty} M_c^n \tilde{f}. \tag{6.29}$$

The series converges because M_c is a contraction on G_R and $M_c^k \tilde{f} \in G_R$. Now,

$$M_c f_* = M_c f + M_c \tilde{f} + \sum_{n=2}^{\infty} M_c^n \tilde{f} = I_c f + I_c \tilde{f} + \sum_{n=1}^{\infty} M_c^n \tilde{f} = f_*. \tag{6.30}$$

since $M_c f = I_c(f + \tilde{f}) = I_c f + I_c \tilde{f}$.

As remarked above, we choose in turn $f(z) = \log z$ and $c = c_1 = \phi_1(1) = -\omega$, $f(z) = \log(z-1)$, $c_2 = \phi_2(0) = \omega$. From (6.29) and noting that $M_c^n g = M^n(g) - (M^n g)(c)$, this gives a fixed point space for M of $< f_1, f_2 >$, where

$$f_1(z) = \log \frac{z}{-\omega} + \sum_{n=0}^{\infty} \sum_{\underline{i} \in I^n} \log \frac{1 + \omega \phi_{\underline{i}} z}{1 + \omega \phi_{\underline{i}}(-\omega)} \tag{6.31}$$

and

$$f_2(z) = \log \frac{z-1}{\omega - 1} + \sum_{n=0}^{\infty} \sum_{\underline{i} \in I^n} \log \frac{1 + \omega \phi_{\underline{i}} z}{1 + \omega \phi_{\underline{i}}(\omega)}, \tag{6.32}$$

where $\underline{i} = (i_1, i_2, \ldots, i_n)$, $I^n = \{1, 2\}^n$, $\phi_{\underline{i}} = \circ_{j=1}^n \phi_{i_j} = \phi_{i_1} \ldots \phi_{i_n}$, for $n > 0$, and the identity map when $n = 0$. We note that f_2 is the fixed point already reported by Mestel *et al* in [46].

A particularly elegant example of a fixed point of M is obtained by putting $f_3 = f_1 - f_2$ to give the fixed point $f_3(z) = \left[\log \frac{z}{z-1} . \frac{\omega - 1}{-\omega} + W\right]$ where $W = \sum_{n=0}^{\infty} \sum_{\underline{i} \in I^n} \log \frac{1 + \omega \phi_{\underline{i}} \omega}{1 + \omega \phi_{\underline{i}}(-\omega)}$. Since this is a fixed point we also have

$$\begin{aligned} f_3 = M f_3 &= M \log \frac{z}{z-1} . \frac{\omega - 1}{-\omega} + 2W \\ &= \log \frac{-\omega z}{-\omega z - 1} . \frac{\omega^2 z + \omega}{\omega^2 z + \omega - 1} . \left(\frac{\omega - 1}{-\omega}\right)^2 + 2W \\ &= \log \frac{z}{z-1} . \omega^2 + 2W \end{aligned}$$

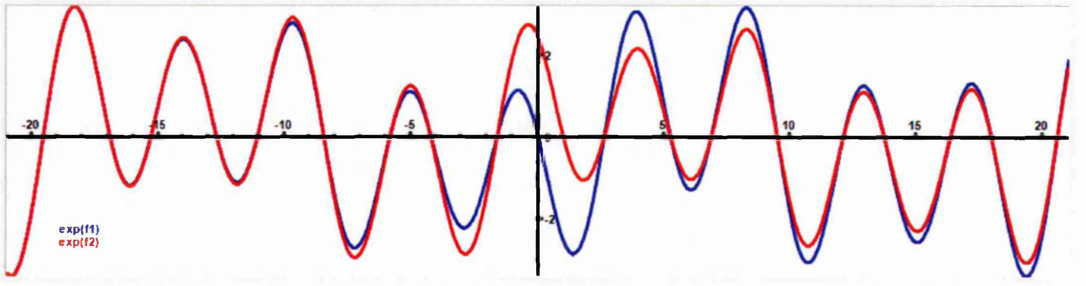

Figure 6.5.1: The real parts of the two multiplicative fixed points

Hence $W = -\log\omega = \log(1+\omega)$, by the properties of ω, and $f_3(z) = \lambda \log\frac{z}{z-1}$, where λ is constant. This gives us the subspace of fixed points $< \log\frac{z}{z-1} >$ and also the identity

$$e^W = \prod_{n=0}^{\infty}\prod_{\underline{i}\in I^n}\left(\frac{1+\omega\phi_{\underline{i}}\omega}{1+\omega\phi_{\underline{i}}(-\omega)}\right) = 1+\omega\,. \tag{6.33}$$

Clearly we can take exponentials of the fixed points f_1, f_2 to obtain instead fixed points of the *multiplicative* functional equation $f(z) = f(-\omega z).f(\omega^2 z+\omega)$. The singularities are removable and can be replaced with zeroes to obtain entire functions. The real parts of $\exp f_1, \exp f_2$ are shown in Figure 6.5.1, and it can be seen that this is consistent with the identity $\frac{\exp f_1}{\exp f_2} = \frac{z}{z-1}$.

Chapter 7

Further research

Quasiperiodic phenomena are intrinsically interesting, but also have a broad range of applications as we have seen. We have studied the Hecke sum, Knill sum, and Sudler product which originated in very different disciplines, and hope we have also shown that unifying their study, and the tools of their study, provides genuine benefits. We used renormalisation techniques to study these quasiperiodic sums and products, and we also studied the renormalisation operators directly, finding the fixed points of the golden renormalisation operator with PESL singularities. However we feel we have done no more than scratch the surface of a fascinating and challenging area of research. In this chapter we will lay out what seem promising directions for further development.

7.1 Quasiperiodic sums and products

Recalling the definition of the quasiperiodic sum $S_n(x, \alpha, f) = \sum_{k=0}^{n-1} f(x + k\alpha)$, there are clearly three distinct directions in which to develop, namely widening the classes of f, x, α respectively. Somewhat surprisingly, from the early work we have done in these directions, it appears that the order of difficulty is the reverse of what might be expected. We list them in the order of that increasing apparent difficulty.

7.1.1 Generalising f

We have focused our studies in this work on important special cases, namely the remainder function, sine, and cotangent. Although these are important in their own right, in applied disciplines we will wish to explore the kinds of perturbations which can be applied to these specific cases without destroying key aspects of the behaviour, and in particular the growth rate characteristics. In the language of renormalisation, we wish to determine their *universality class*.

Our initial analysis suggests that that these particular functions are generic in the sense that they can be substantially distorted without affecting the asymptotic behaviour. The most important constraints seem to be the conservation of the function's average, concavity, and either a symmetry or anti-symmetry about some point of the circle.

7.1.2 Generalising α

It is common practice when working with properties of irrationals to start with the golden ratio as it is the simplest to work with. One hopes to develop understanding and techniques from this simplest example which can be applied to arithmetically more complex examples. There is a simple generalisation to be made immediately, from the golden rotation to the other numbers which have continued fractions of period 1, but this is a very sparse set. The next logical step is to continued fractions of longer periods: this brings a step change in complexity, but looks feasible. This provides us with a class which is at least dense in the continuum, but unfortunately is still of measure 0. The next step would be to study eventually recurring continued fractions (representing all quadratic irrationals). We have not yet had opportunity to investigate this, but we suspect that this might not be too difficult. However this is still a class of measure 0: the untimate goal would be to break through to a class of positive measure. This looks more difficult and will probably need some additional techniques.

A generalisation away from rigid rotations is probably also possible. We expect that the various classical results of Denjoy, Herman and Yoccoz on the conjugacy of rigid rotations with circle diffeomorphisms will provide a route for broadening results on quasiperiodic sums and products into more general classes of sums and products (eg of the form $S_n(x, \alpha, f) = \sum_{k=0}^{n-1} f(T^k x)$ for some diffeomorphic map T). The perturbation of rigid rotations could also be worth exploring using KAM techniques.

7.1.3 Generalising x

In the case of the Hecke sum, we were able to generalise existing results for $x = \alpha$ to hold for any x. However in the case of the Knill sum and Sudler product, we studied only the special case $x = \alpha$. It is simple to generalise this to the set of values $x = \{n\alpha\}$ for $n \in \mathbb{Z}$ which is dense in the circle, but more general values of x are more problematic. This may seem slightly surprising given that we are working with value functions which are piecewise smooth, and where one might expect to be able to construct the full set of values from a known dense set. The problem is that the quasiperiodic sums and products develop increasing sensitivity to initial conditions with n, and for any fixed $\epsilon > 0$ the variation of the sum or product between x and $x + \epsilon$ may be unbounded as n grows. In this respect the Knill sum is more promising than the Sudler product as the variation is much more controlled, and it seems likely that we could obtain results which would apply for almost all x.

Even in the case of the Sudler product, we do have a dense set of points which we understand, and there seems some promise in applying shadowing techniques to approximate the orbit of a general point via a sequence of orbit segments of close known points. What we do not yet know is the size of the class of general points for which this might provide useful results.

7.2 Renormalisation and associated operators

In chapter 6 we introduced a new theory of non-zero fixed points of linear operators T, showing the existence of a fixed point operator $\widehat{T}$ derived from T whose image is all the fixed points of T. In order to have a non-trivial theory, we found the need to work with spaces of unbounded functions. This is a rich and interesting theory in its own right and deserving of further development.

In chapter 6 we introduced and focused on PESL spaces (containing complex analytic functions with poles, essential singularities, and simple logarithmic singularities - see (??) for the full definition). We chose these as important initial examples of unbounded functions. However it would be interesting to see how far our methods would extend to spaces of functions with other types of unbounded singularities such as algebraic or more complex logarithmic singularities.

We developed an initial theory for affine CSOs (composition sum operators), a class of operators whose fixed points are important in the study of the renormalisation of a variety of physical problems. In particular this has enabled us, under a simple set of constraints, to develop tests for the existence of fixed points in PESL spaces, and to exhibit their necessary form. There are several directions for further research in this area as discussed in our paper[58].

First, we may extend our study to cover the full spectrum of affine CSOs. Considered as linear operators on function spaces of analytic functions, CSOs are compact operators and thus have discrete non-zero spectrum. It is likely that the techniques developed in chapter 6 may be adapted to construct more general eigenfunctions of affine CSOs, with a view to obtaining a full description of their spectra.

Second, it is likely that the approach of, for example, [45] may be applied to understand fixed points of an affine CSO with non-simple unbounded singularity set and, more generally, all periodic points of an affine CSO. A full understanding of the latter is indeed necessary for a complete description of all the fixed points of a CSO. For let f_1, f_2 be a periodic orbit of period-2 of a CSO T. Then $Tf_1 = f_2$ and $Tf_2 = f_1$ so that $f = f_1 + f_2$ is generally a non-simple fixed point of T, a construction that clearly generalises to other periods.

Third, an important future direction is to consider more general CSOs than affine CSOs. Of course, explicit construction of fixed points (and more general eigenfunctions) may not be in general possible for non-affine CSOs. However a general theory may well be possible and it may be possible to make extensive progress for special important cases. An analogy may be drawn here with the theory of linear differential equations. The theory of constant coefficient linear differential equations is complete, while that for general linear equations is less well developed except in special cases of particular interest. Nevertheless, non-constant coefficient CSOs are of considerable interest. For example, the full stretch-fold-shear toy model studied by Gilbert [18, 19] involves a study of the spectrum of the CSO T on complex-valued functions c of a real variable x given by

$$Tc(x) = e^{i\alpha(x-1)/2}\, c\left(\frac{x-1}{2}\right) - e^{i\alpha(1-x)/2}\, c\left(\frac{1-x}{2}\right), \tag{7.1}$$

where $\alpha \geq 0$ is a real parameter, corresponding to the level of shear in the map.

Recall that a CSO given by (6.2) is affine if each of the coefficients a_i is constant and each of the maps α_i is an affine contraction. While it would certainly be interesting to relax each of these conditions, a theory for non-constant a_i would be of immediate application is several areas including the kinematic dynamo theory discussed in [18, 19] and in the study of strange non-chaotic attractors [15].

Finally, as alluded to above, CSOs may have more complex attractors than fixed points, and in particular the study of periodic cycles is an obvious generalisation. These do not necessarily require unbounded functions spaces, and indeed the operator M of chapter 6 has already been studied in spaces of piecewise constant real functions, and has found fruitful application in several fields (again see chapter 6 for details). It would be very interesting to develop a general theory of CSOs for spaces of functions with discontinuities either of the function or its derivatives.

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